

THE EULER EQUATIONS IN PLANAR NONSMOOTH CONVEX DOMAINS

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ABSTRACT. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. We consider the Euler system

$$(P) \quad \begin{cases} \partial_t \mathbf{u}(x, t) + (\mathbf{u} \cdot \nabla) \mathbf{u}(x, t) + \nabla \pi(x, t) = \mathbf{f}(x, t), & x \in \Omega, t \in (0, T), \\ \nabla \cdot \mathbf{u}(x, t) = 0 & x \in \Omega, t \in (0, T), \end{cases}$$

with impermeability boundary conditions. For $\frac{4}{3} \leq p \leq 2$, divergence-free initial data $\mathbf{u}_0 = \mathbf{u}(t=0) \in \mathbf{W}^{1,p}(\Omega)$ with zero normal component on $\partial\Omega$, and forcing term $\mathbf{f} \in L^p(0, T; \mathbf{W}^{1,p}(\Omega))$, we show the existence of a pair

$$\mathbf{u} = \mathbf{u}(x, t) \in L^\infty(0, T; \mathbf{W}^{1,p}(\Omega)) \cap \mathcal{C}([0, T]; L^2(\Omega)), \quad \pi \in L^p(0, T; W^{1,s(p)}(\Omega)),$$

which solves the Euler system (P) almost everywhere on $\Omega \times (0, T)$. This enriches and extends the results of Taylor [31].

In the physically interesting case of a rectangle $\Omega = [0, L_1] \times [0, L_2]$, a similar result holds for all $2 < p < \infty$ as well. Moreover, if

$$\operatorname{curl} \mathbf{u}_0 \in L^\infty(\Omega), \quad \mathbf{f} \in L^\infty(0, T; \mathbf{W}^{1,\infty}(\Omega))$$

the solution \mathbf{u} is unique and $\nabla \mathbf{u} \in L^\infty(0, T; \mathbf{bmo}_z(\Omega)^{2 \times 2})$. This is a consequence of a new $\mathbf{bmo}_z(\Omega)$ regularity estimate for the Dirichlet problem on a rectangle.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set. We are concerned with the Euler equations, describing the motion of a perfect inviscid fluid inside Ω ,

$$(P) \quad \begin{cases} \partial_t \mathbf{u}(x, t) + (\mathbf{u} \cdot \nabla) \mathbf{u}(x, t) + \nabla \pi(x, t) = \mathbf{f}(x, t), & x \in \Omega, t \in (0, T), \\ \nabla \cdot \mathbf{u}(x, t) = 0, & x \in \Omega, t \in (0, T), \end{cases}$$

where $\mathbf{u}(x, t) = (u_1(x_1, x_2, t), u_2(x_1, x_2, t))$ stands for the velocity of the fluid particle located at x at time t , $\nabla \pi(x, t)$ stands for the pressure gradient, and $\mathbf{f} = \mathbf{f}(x, t)$ is an external forcing term. We endow (P) with initial and boundary conditions

$$(1.1) \quad \mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad x \in \Omega,$$

$$(1.2) \quad \mathbf{u}(x, t) \cdot \mathbf{n}(x) = 0, \quad x \in \partial\Omega, t \in (0, T);$$

we remark that the impermeability boundary condition (1.2) has to be properly reformulated when $\partial\Omega$ is not regular enough to admit a normal vector \mathbf{n} almost everywhere.

This work is primarily motivated by the study of the well-posedness of the barotropic mode of the inviscid primitive equations of the atmosphere and the oceans [27]. As explained in [8, 9, 27], a certain vertical modal expansion of the primitive equations leads to an infinite system of coupled barotropic - baroclinic modes. In a first approximation, one

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can neglect the baroclinic modes and we obtain for the barotropic mode a system of equations very similar to the two-dimensional inviscid Euler equations in a rectangle. Henceforth this system is called the barotropic system. The study of the well-posedness of the barotropic system is thus very similar to the study of the well-posedness of the (inviscid) incompressible Euler equations in a rectangle Ω ; complementary developments regarding this very problem will appear elsewhere.

The study of the well-posedness of the Euler equations of incompressible fluid has a long history regarding both weak and strong solutions, starting with [24, 36], the latter article of Wolibner containing the first proof of global in time existence and uniqueness of regular solutions in two dimensions. This result was simplified and extended by Kato in [22]; when Ω is sufficiently smooth, with some additional work, one obtains up to \mathcal{C}^∞ solutions, see e. g. [32, 33].

The notion of weak solution to the two-dimensional Euler equations has been introduced by Yudovich [37]. Yudovich, and later Bardos with a different (vanishing viscosity) approach [2] consider initial data with L^p vorticity, and show existence of weak solutions and, in the case of bounded initial vorticity, uniqueness. Among many references on the well-posedness of the incompressible Euler equations in a bounded smooth domain, let us quote the classical articles [2, 5, 38] (see also [32, 33]). Other weaker notions of solutions (e. g. initial datum in $L^2(\Omega)$, with no assumption on the initial vorticity) have been considered by several authors. See for instance [12, 28, 29] and Remark 6.1 below.

In view of the lack of smoothness of Ω , we focus on the notion of weak solutions with L^p vorticity. Our main goal is to establish existence of weak solutions to (P) set in the non-smooth domain $\Omega = [0, L_1] \times [0, L_2]$ assuming that the initial vorticity is in L^p , and that in addition uniqueness of solutions holds in the case of bounded initial vorticity.

However, for the construction of a solution, we work in the more general case of a bounded convex domain Ω , as in the previous, related article [31], where, for $\frac{12}{7} < p \leq 2$, given a divergence-free initial datum \mathbf{u}_0 with $\text{curl } \mathbf{u}_0 \in L^p(\Omega)$, an $L^\infty(0, T; \mathbf{W}^{1,p}(\Omega))$ weak solution is constructed as the limit of smooth solutions to (P) set on smooth convex subdomains Ω_n increasing to Ω . This principle has also been used in [17] for more general domains (complements of a finite number of compact connected sets with positive Sobolev capacity), albeit the notion of solution considered there is significantly weaker.

We follow the same principle, but make a more systematic study of the approximated problems (P_n) . In particular we impose a uniform Lipschitz character to the domains Ω_n . This uniformity reflects on the uniform boundedness of the Leray projectors associated to each subdomain, which we exploit in our compactness arguments. As a result, our solution is slightly more regular than the one constructed in [31], being continuous in time with values in $L^2(\Omega)$ and belonging to $\mathbf{W}^{1,p}(\Omega)$ for each time $t \in [0, T]$, and satisfying (P) almost everywhere in $\Omega \times (0, T)$. The range of allowed exponents is $p \in [\frac{4}{3}, 2]$. The restriction $p \geq \frac{4}{3}$ can be lifted if one is interested in a weaker notion of solution than the one given in Subsection 4.1.2. The restriction $1 < p \leq 2$ corresponds to the sharp range of exponents for the regularity of the Biot-Savart law (see Theorem 2.2), giving the gradient of the velocity in terms of the vorticity, in a general convex domain, and seems unavoidable if one aims for a *reversible* existence result (in the sense that $\mathbf{u}(t)$ for $t > 0$ belongs to the same space where the data $\mathbf{u}(0)$

is required to be, so that we can solve the backward Euler equations with initial data $\mathbf{u}(t)$. The construction is summarized in Proposition 4.1 and Theorem 5.1.

Theorem 5.2 deals with (P) on the rectangle $\Omega = [0, L_1] \times [0, L_2]$. It contains the extension of the existence theorem 5.1 to the range $p \in (2, \infty]$, as well as the main result of uniqueness of solutions in the case $p = \infty$. To the best of our knowledge, this is the first uniqueness result for Euler solutions on a domain with corners without requiring additional assumptions on the initial vorticity other than being bounded. In [23], the author proves a uniqueness result for a class of domains somehow complementary to ours, that is domains with a finite number of corners having angles greater than $\pi/2$, assuming that the initial vorticity is bounded and has definite signum. Our proof follows Yudovich's energy method, and relies on the endpoint $L^\infty(\Omega) \rightarrow \mathbf{W}^{2,\text{bmo}}(\Omega)$ regularity result for the solution to the Dirichlet problem on a rectangle, Proposition 3.1, which appears to be new, to the best of our knowledge. We do not dwell on the latter point, but unbounded initial vorticities with loglog-type blowup of the L^p -norms as $p \rightarrow \infty$, like in [38], would also suffice for uniqueness.

Theorem 5.2 holds *verbatim* for the more general class of domains, that is bounded domains with piecewise smooth boundary and with corners of aperture of the form $\alpha = \frac{\pi}{m}$, for some integer $m \geq 2$. We briefly discuss this extension in Subsection 6.3.

Plan of the paper. In Section 2, we develop the necessary tools for the analysis of (P) on a bounded convex non-smooth domain Ω . Section 3 contains the endpoint-type regularity result for the Dirichlet problem on a rectangle, which will be instrumental in establishing uniqueness of solutions. In Section 4 we give a weak formulation of (P), and construct a weak solution to (P) on a bounded convex domain (Proposition 4.1). Section 5 contains the statements and the proofs of the main results, Theorems 5.1 and 5.2. In Section 6, we make some additional remarks: in particular, we briefly outline, for comparison's sake, the analogous of Theorem 5.2 in the space-periodic case $\Omega = \mathbb{T}^2$. We also discuss some extensions of Theorem 5.2 to a more general class of domains with corners.

Notation. Given a domain $\Omega \subset \mathbb{R}^2$, and scalar functions $u, v : \Omega \rightarrow \mathbb{R}$, vector valued functions $\mathbf{u}, \mathbf{v} : \Omega \rightarrow \mathbb{R}^2$, we denote

$$(u, v)_\Omega := \int_\Omega u v, \quad (\mathbf{u}, \mathbf{v})_\Omega := \int_\Omega \mathbf{u} \cdot \mathbf{v}.$$

Throughout, for $p \in [1, \infty]$, we use the notations

$$p' = \frac{p}{p-1}, \quad p^* = \begin{cases} \frac{2p}{2-p} & 1 \leq p < 2 \\ \infty & p \geq 2 \end{cases}$$

respectively for the Hölder and Sobolev conjugate exponents of p .

Bump functions. Let $d \geq 1$, and $\phi^d : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth nonnegative radial function supported in $\{|x| \leq \frac{1}{2}\}$, equal to 1 in $\{|x| \leq \frac{1}{4}\}$, and with $\int_{\mathbb{R}^d} \phi^d = 1$. We will make use of the functions

$$\phi_\varepsilon^d(x) = (\text{Dil}_\varepsilon^1 \phi^d)(x) := \varepsilon^{-d} \phi(\varepsilon^{-1}x), \quad \varepsilon > 0,$$

and refer to them as L^1 -bump functions (or L^1 -bumps). Indeed, note that $\int_{\mathbb{R}^d} \phi_\varepsilon^d(x) dx = 1$.

2. ELLIPTIC REGULARITY IN A BOUNDED CONVEX DOMAIN

In this section, we set the foundation for our analysis of the Euler system (P). Throughout the section, Ω is an open, bounded, convex subset of \mathbb{R}^2 which contains the origin. In Section 3, we will specialize to the case of a rectangular domain and develop further elliptic regularity results.

We first recall the analytic properties of the boundary $\partial\Omega$ and construct an approximation of Ω by an increasing sequence of convex smooth subdomains with uniformly Lipschitz boundary. Then, we describe the normal trace operator on Ω , introduce the class of tangential vector fields, and establish the Helmholtz decomposition of $\mathbf{L}^p(\Omega)$, for $1 < p < \infty$. Finally, we discuss some regularity results for the Dirichlet problem in Ω , which we exploit to define the spaces in which the evolution of the Euler system (P) will take place.

2.1. Regularity and approximation of bounded convex domains. We begin with a proposition.

Proposition 2.1. *Let Ω be a bounded convex open set containing the origin. There exist positive constants M_Ω, δ_Ω , and a finite collection of open squares $\{Q_i : i = 1, \dots, N_\Omega\}$ of diameter δ such that:*

- $\bar{\Omega} \subset \bigcup_{i=1}^{N_\Omega} Q_i$, $Q_i = c(Q_i) + Q_0$, $Q_0 = \{|y_1|, |y_2| < \delta_\Omega\}$,
- whenever $Q_i \cap \partial\Omega \neq \emptyset$, there exists a function $\beta_i : (-\delta_\Omega, \delta_\Omega) \rightarrow \mathbb{R}$, in the coordinates with origin the center of Q_i and oriented along the sides of Q_i , with the properties
 - β_i is convex and Lipschitz with constant M_Ω ,
 - $Q_i \cap \partial\Omega = \{(y_1, \beta_i(y_1)) : y_1 \in (-\delta_\Omega, \delta_\Omega)\}$,
 - $Q_i \cap \Omega = \{(y_1, y_2) \in Q_i : y_2 > \beta_i(y_1)\}$,
- Ω has the strong local Lipschitz property with constants $M_\Omega, \delta_\Omega, N_\Omega$.

Proof. It is known (see [19, Corollary 1.2.2.3] that Ω bounded and convex implies that Ω has a Lipschitz boundary (in the sense of [19, Definition 1.2.2.1]), which is exactly what is described in the first two assertions. The fact that β_i is convex is a consequence of the fact that its epigraph $Q_i \cap \Omega$ is convex. Finally, the first two assertions imply the strong local Lipschitz property with constants $M_\Omega, \delta_\Omega, N_\Omega$ as described in [1, IV.4.2]. \square

2.1.1. Approximation by smooth convex subdomains. We construct a sequence of smooth convex domains Ω_n increasing to Ω , that is

$$(2.1) \quad \Omega_n \text{ smooth convex}, \quad \Omega_n \Subset \Omega_{n+1} \Subset \dots \Subset \Omega, \quad \Omega = \bigcup_n \Omega_n.$$

and with the property that

$$(2.2) \quad \text{the constants in the strong local Lipschitz property of } \Omega \text{ are uniform in } n.$$

We introduce the Minkowski functional of Ω

$$\mu_\Omega : \bar{\Omega} \rightarrow [0, \infty), \quad \mu_\Omega(x) = \inf \{\lambda > 0 : \lambda^{-1}x \in \bar{\Omega}\}.$$

The function μ_Ω is a convex function on $\bar{\Omega}$ (see [21, pp. 57-59]). A convex function on a compact subset of any normed space is globally Lipschitz (see [26] for a simple proof): thus,

call L_Ω the Lipschitz constant of μ_Ω . The function $\rho = \mu_\Omega - 1$ is convex and Lipschitz with the same constant L_Ω , and

$$\Omega = \{x \in \bar{\Omega} : \rho(x) < 0\}, \quad \partial\Omega = \{x \in \bar{\Omega} : \rho(x) = 0\}$$

For $\varepsilon > 0$, let Ω^ε be the ε -neighborhood of Ω . It is easy to verify that the mollification (see Section 1 for notation)

$$\rho_\varepsilon : \Omega^\varepsilon \rightarrow [-1, 0], \quad \rho_\varepsilon := \rho * \phi_\varepsilon^2$$

is smooth and convex, and moreover that the Lipschitz constants of $\{\rho_\varepsilon : \varepsilon > 0\}$ are uniformly bounded by L_Ω . Finally, we have that, for $\varepsilon \rightarrow 0$, $\rho_\varepsilon \rightarrow \rho$ in the uniform Lipschitz norm, that is

$$\sup_{x, y \in \Omega} \frac{|\rho_\varepsilon(x) - \rho(x) - \rho_\varepsilon(y) + \rho(y)|}{|x - y|} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

Choose a subsequence n_k with

$$\sup_{x \in \Omega} \left| \rho_{\frac{1}{n_k}}(x) - \rho_{\frac{1}{n_{k+1}}}(x) \right| < (n_k)^{-3},$$

and define

$$\Omega_k := \left\{ x \in \Omega : \rho_{\frac{1}{n_k}}(x) < -\frac{1}{n_k} \right\};$$

it follows that Ω_k is a convex open set, $\Omega_k \Subset \Omega_{k+1} \Subset \Omega$, and that $\bigcup \Omega_k = \Omega$. Moreover the $\partial\Omega_k$ are smooth, and uniformly Lipschitz (with respect to k). Thus each Ω_k has the strong local Lipschitz property with Lipschitz constant uniformly bounded in k . The construction (2.2) is thus completed.

Several important consequences of (2.1)-(2.2) will be derived in the next subsections. Here, we mention that (2.2) guarantees that the implicit constants appearing in the Sobolev embeddings and trace theorems on Ω_n , which depend on the constants in the strong local Lipschitz property (see Theorem IV.4.1 and its proof in [1] for instance) are uniform in n (they do depend on Ω however, through $M_\Omega, \delta_\Omega, N_\Omega$ and L_Ω).

2.2. Normal vector, normal traces, tangential vector fields. Maintaining the notation of Proposition 2.1, we observe that for each i , $\beta'_i(y_1)$ is defined almost everywhere on $(-\delta_\Omega, \delta_\Omega)$ and $|\beta'_i(y_1)| \leq M_\delta$ wherever defined. We can thus define the normal vector almost everywhere on $Q_i \cap \partial\Omega$ by

$$\mathbf{n}(x) = \mathbf{n}(y_1, \beta_i(y_1)) := \frac{(-\beta'_i(y_1), 1)}{\sqrt{1 + (\beta'_i(y_1))^2}}, \quad x = (y_1, \beta_i(y_1)) \in Q_i \cap \partial\Omega.$$

This definition can be extended to all of Q_i by $\mathbf{n}(y_1, y_2) = \mathbf{n}(y_1, \beta_i(y_1))$, and using a partition of unity subordinated to the covering of $\bar{\Omega}$ by the Q_i , to a bounded vector field on all of $\bar{\Omega}$.

Thus, for $\mathbf{v} \in \mathcal{D}(\bar{\Omega}; \mathbb{R}^2)$,

$$\gamma_{\mathbf{n}} \mathbf{v}(x) := \mathbf{v}(x) \cdot \mathbf{n}(x), \quad \gamma_{\mathbf{t}} \mathbf{v}(x) := \mathbf{v}^\perp(x) \cdot \mathbf{n}(x), \quad x \in \partial\Omega$$

is defined almost everywhere on $\partial\Omega$. We are interested in the spaces

$$\mathbf{L}_{\text{div}}^p(\Omega) = \{\mathbf{v} \in \mathbf{L}^p(\Omega) : \text{div } \mathbf{v} \in L^p(\Omega)\}, \quad 1 \leq p \leq \infty.$$

The classical $W^{1,p}(\Omega) \rightarrow W^{1-\frac{1}{q},q}(\partial\Omega)$ trace theorem due to Gagliardo [15] yields the lemma below, arguing along the same lines of [34, Theorem I.1.2]. For a definition of Sobolev spaces (of fractional order) on Lipschitz submanifolds of \mathbb{R}^d , see Subsection 1.3.3 in [19]. Note that

if $f \in \mathcal{D}(\bar{\Omega})$ (that is, f is restriction to $\bar{\Omega}$ of a function in $\mathcal{C}^\infty(\mathbb{R}^2)$), then the restriction of f to $\partial\Omega$ is a Lipschitz function on the Lipschitz submanifold $\partial\Omega$.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain and $p \in (1, \infty)$. The map $\gamma_n : \mathbf{v} \in \mathcal{D}(\bar{\Omega}; \mathbb{R}^2) \rightarrow \gamma_n \mathbf{v} \in \text{Lip}(\partial\Omega)$ extends as a linear bounded map*

$$\gamma_n : \mathbf{L}_{\text{div}}^p(\Omega) \rightarrow W^{-\frac{1}{p}, p'}(\partial\Omega)$$

and the generalized Stokes formula

$$(2.3) \quad (\mathbf{v}, \nabla \varphi)_\Omega + (\text{div } \mathbf{v}, \varphi)_\Omega = \langle \gamma_n \mathbf{v}, \gamma_0 \varphi \rangle$$

holds for every $\mathbf{v} \in \mathbf{L}_{\text{div}}^p(\Omega)$ and $\varphi \in W^{1, p'}(\Omega)$.

Remark 2.1. If Ω_n is an approximating sequence of domains as in Subsection 2.1.1, the norms of the $\gamma_n \in \mathcal{L}(\mathbf{L}_{\text{div}}^p(\Omega)) \rightarrow W^{-1/p, p'}(\partial\Omega_n)$ are uniformly bounded in n . This uniformity descends from the uniformity of the constants in the Gagliardo trace theorem, [15].

2.2.1. L^p -tangential vector fields. We say that $\mathbf{v} \in \mathbf{L}^1(\Omega)$ is a tangential divergence free vector field if

$$(2.4) \quad (\mathbf{v}, \nabla \varphi)_\Omega = 0 \quad \forall \varphi \in \mathcal{D}(\bar{\Omega}).$$

As a consequence of Lemma 2.1, it follows that

$$(2.5) \quad \mathbf{L}_\tau^p(\Omega) := \{\mathbf{v} \in \mathbf{L}^p(\Omega) : (2.4) \text{ holds}\} = \{\mathbf{v} \in \mathbf{L}_{\text{div}}^p(\Omega) : \text{div } \mathbf{v} = 0, \gamma_n \mathbf{v} = 0\}.$$

2.3. Helmholtz decomposition. Let $\mathcal{V} := \{\mathbf{v} \in \mathcal{D}(\Omega; \mathbb{R}^2) : \text{div } \mathbf{v} = 0\}$. It is well known (see for example Theorem I.1.4 in [34]) that for Ω Lipschitz

$$(2.6) \quad \mathbf{L}_\tau^2(\Omega) = \text{the closure of } \mathcal{V} \text{ in } \mathbf{L}^2(\Omega), \quad \mathbf{L}_\tau^2(\Omega)^\perp = \{\nabla \pi : \pi \in H^1(\Omega)\}.$$

Let us denote by $P_\Omega : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}_\tau^2(\Omega)$ the corresponding orthogonal projector. The following result, due to Geng and Shen [16] allows us to obtain (2.6) for all $1 < p < \infty$ and extend P_Ω boundedly to $\mathbf{L}^p(\Omega)$. Note that Theorem I.1.4 in [34] (i.e. the case $p = 2$ of Theorem 2.1 below) holds whenever Ω is Lipschitz; actually, the range $p \in (\frac{3}{2} - \varepsilon, 3 + \varepsilon)$ is known to be sharp for general Lipschitz domains [13].

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain and $1 < p < \infty$. The following hold:*

$$(2.7) \quad \mathbf{L}_\tau^p(\Omega) = \text{the closure of } \mathcal{V} \text{ in } \mathbf{L}^p(\Omega);$$

P_Ω extends to a bounded linear operator

$$P_\Omega : \mathbf{L}^p(\Omega) \rightarrow \mathbf{L}_\tau^p(\Omega), \quad P_\Omega|_{\mathbf{L}_\tau^p(\Omega)} = \text{Id}_{\mathbf{L}_\tau^p(\Omega)},$$

with operator norm only depending on M_Ω , δ_Ω , N_Ω , L_Ω . Moreover, for each $\mathbf{v} \in \mathbf{L}^p(\Omega)$, there exists $\pi \in W^{1, p}(\Omega)$, unique up to an additive constant such that

$$(2.8) \quad P_\Omega^\perp \mathbf{v} := (I - P_\Omega) \mathbf{v} = \nabla \pi.$$

Proof. The second and third statements appear almost verbatim in [16, Theorem 1.3]. Let us show how they imply (2.7); denote by \mathbf{H}_τ^p the space on the right hand side of (2.7). The backward inclusion is easy (see the proof of [34, Theorem I.1.4] for example). To get equality, we begin by characterizing the annihilator of \mathbf{H}_τ^p in $\mathbf{L}_\tau^p(\Omega)$ as

$$(2.9) \quad (\mathbf{H}_\tau^p)^\perp := \{F \in \mathcal{L}(\mathbf{L}_\tau^p(\Omega), \mathbb{R}) : F|_{\mathbf{H}_\tau^p} = 0\} = \nabla W^{1, p'}(\Omega), \quad \nabla W^{1, p'}(\Omega) := \{\nabla \pi : \pi \in W^{1, p'}(\Omega)\}.$$

Proof of (2.9). It is known (see [35] for a simple proof) that

$$F \in \mathcal{D}'(\Omega; \mathbb{R}^2), \quad F(\mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathcal{V} \implies F = \nabla \pi, \quad \pi \in \mathcal{D}'(\Omega).$$

Now from the Riesz representation theorem, for each $F \in \mathcal{L}(\mathbf{L}_\tau^p(\Omega), \mathbb{R})$ there exists a (possibly nonunique) $\mathbf{f} \in \mathbf{L}^{p'}(\Omega)$ such that $F(\mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega$ for all $\mathbf{v} \in \mathbf{L}_\tau^p(\Omega)$. From the above characterization, it follows that $\mathbf{f} = \nabla \pi$, and thus $\pi \in W^{1,p'}(\Omega)$. \square

With (2.9) in hand, we show that $(\mathbf{H}_\tau^p)^\perp = \{\mathbf{0}\}$, thus proving (2.7). Let $F \in (\mathbf{H}_\tau^p)^\perp$, and $\mathbf{f} \in \mathbf{L}^{p'}(\Omega)$ as above. By (2.8), it follows that $\mathbf{f} = (I - P_\Omega)\mathbf{f}$, i.e. $P_\Omega \mathbf{f} = 0$. Therefore

$$(\mathbf{f}, \mathbf{v})_\Omega = (\mathbf{f}, P_\Omega \mathbf{v})_\Omega = (P_\Omega \mathbf{f}, \mathbf{v})_\Omega = 0 \quad \forall \mathbf{v} \in \mathbf{L}_\tau^p(\Omega),$$

and this proves the last claim. Here we used the same notation (with slight abuse) for both $P_\Omega : \mathbf{L}^p(\Omega) \rightarrow \mathbf{L}^p(\Omega)$ and $P_\Omega : \mathbf{L}^{p'}(\Omega) \rightarrow \mathbf{L}^{p'}(\Omega)$. \square

2.4. The Dirichlet problem and the Biot-Savart law. Denote by

$$A_\Omega = -\Delta \text{ on } \Omega \text{ with Dirichlet boundary conditions,} \quad \text{dom}(A_\Omega) = H^2(\Omega) \cap H_0^1(\Omega).$$

We recall the following consequence of the Lax-Milgram lemma.

Proposition 2.2. *Let $f \in H^{-1}(\Omega)$ be given. Then there exists a unique $F \in H_0^1(\Omega)$ satisfying*

$$(2.10) \quad A_\Omega F = f, \quad \|F\|_{H_0^1(\Omega)} \leq \|f\|_{H^{-1}(\Omega)}.$$

Referring to (2.10), we use the notation $F = G_\Omega f$. The classical theory of elliptic equations (see for example [18]) tells us that, when Ω is a bounded smooth domain, and $f \in L^p(\Omega)$, $G_\Omega f$ has two derivatives in $L^p(\Omega)$ for any $1 < p < \infty$. This is no longer true in general if the domain Ω is merely bounded and convex. However, the above result still holds in the range $1 < p \leq 2$: we state this precisely in the theorem below, due to Fromm [14].

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. Let $1 < p \leq 2$ and $f \in L^p(\Omega)$ be given. Then, there exists a positive constant $C_{p,\Omega}$, depending only on p and on the Lipschitz character of Ω , such that*

$$(2.11) \quad G_\Omega f \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad \|G_\Omega f\|_{W^{2,p}(\Omega)} \leq C_{p,\Omega} \|f\|_{L^p(\Omega)}.$$

Let Ω_n be the approximating sequence to Ω constructed in (2.1). In this context, for a function $F : \Omega_n \rightarrow \mathbb{R}$, we denote by \tilde{F} its extension by zero to Ω . Note that

$$(2.12) \quad F \in W_0^{1,p}(\Omega_n) \implies \tilde{F} \in W_0^{1,p}(\Omega) \quad \text{and} \quad \|\tilde{F}\|_{W_0^{1,p}(\Omega)} = \|F\|_{W_0^{1,p}(\Omega_n)}.$$

We state and prove a so-called Γ -convergence type result. The restrictions $f_n, f \in L_{\text{loc}}^1(\Omega)$ can be lifted, but we only need the particular case contained in the lemma below.

Lemma 2.2. *Let $f_n, f \in L_{\text{loc}}^1(\Omega)$ and set $(f_n)_{\Omega_n} = f_n \mathbf{1}_{\Omega_n}$. Then*

$$(2.13) \quad f_n \rightarrow f \text{ in } H^{-1}(\Omega) \implies \widetilde{G_{\Omega_n}(f_n)_{\Omega_n}} \rightarrow G_\Omega f \text{ weakly in } H_0^1(\Omega).$$

Proof. By density it suffices to show that

$$\lim_{n \rightarrow \infty} (\nabla \widetilde{G_{\Omega_n}(f_n)_{\Omega_n}} - \nabla G_\Omega f, \nabla \varphi)_\Omega = 0 \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Fix $\varphi \in \mathcal{D}(\Omega)$ and choose N such that $\text{supp } \varphi := D \Subset \Omega_N$. Then for $n \geq N$, $\text{supp } \varphi \subset \Omega_n$ and

$$\begin{aligned} (\nabla \widetilde{G_{\Omega_n}(f_n)_{\Omega_n}}, \nabla \varphi)_\Omega &= (\nabla \widetilde{G_{\Omega_n}(f_n)_{\Omega_n}}, \nabla \varphi)_{\Omega_n} = (\nabla G_{\Omega_n}(f_n)_{\Omega_n}, \nabla \varphi)_{\Omega_n} \\ &= -(\Delta G_{\Omega_n}(f_n)_{\Omega_n}, \varphi)_{\Omega_n} \\ &= -(f_n, \varphi)_{\Omega_n} = -(f_n, \varphi)_\Omega, \end{aligned}$$

and similarly $(\nabla G_\Omega f, \nabla \varphi)_\Omega = -(f, \varphi)_\Omega$. Thus

$$(\nabla \widetilde{G_{\Omega_n}(f_n)_{\Omega_n}} - \nabla G_\Omega f, \nabla \varphi)_\Omega = (f_n - f, \varphi)_\Omega = \langle f_n - f, \varphi \rangle \rightarrow 0,$$

and the lemma follows. \square

We now make the connection between the Euler system and A_Ω more explicit. Set

$$(2.14) \quad \mathbf{K}_\Omega f := \nabla^\perp(G_\Omega f), \quad f \in H^{-1}(\Omega).$$

We have that

$$(2.15) \quad \mathbf{K}_\Omega \in \mathcal{L}(H^{-1}(\Omega), \mathbf{L}_\tau^2(\Omega)),$$

$$(2.16) \quad \mathbf{K}_\Omega \in \mathcal{L}(L^p(\Omega), \mathbf{W}^{1,p}(\Omega)) \cap \mathbf{L}_\tau^2(\Omega), \quad 1 < p \leq 2.$$

Proof of (2.15)-(2.16). Due to (2.11) and (2.10), we are only left to verify that

$$(2.17) \quad \text{div } \mathbf{K}_\Omega f = 0, \quad \gamma_n \mathbf{K}_\Omega f = 0, \quad \forall f \in H^{-1}(\Omega).$$

Let $\varphi \in \mathcal{D}(\overline{\Omega})$. We then have, integrating by parts,

$$(\mathbf{K}_\Omega f, \nabla \varphi)_\Omega = (\nabla(G_\Omega f), \nabla^\perp \varphi)_\Omega = -(G_\Omega f, \text{div } \nabla^\perp \varphi)_\Omega + \int_{\partial\Omega} (G_\Omega f) \nabla^\perp \varphi \cdot \mathbf{n},$$

and both terms vanish in the right-hand side. We conclude by means of (2.7). \square

The next lemma shows that curl is the left inverse of $\nabla^\perp \circ G_\Omega$ on $L^p(\Omega)$, $1 < p \leq 2$.

Lemma 2.3. *Let $1 < p \leq 2$ and $f \in L^p(\Omega)$. Then*

$$(2.18) \quad f = \text{curl } \mathbf{K}_\Omega f.$$

Proof. By density, it suffices to show that

$$(2.19) \quad (f, \varphi)_\Omega = (\text{curl } \mathbf{K}_\Omega f, \varphi), \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Let now $\varphi \in \mathcal{D}(\Omega)$. We have

$$\begin{aligned} (f, \varphi)_\Omega &= (-\Delta G_\Omega f, \varphi)_\Omega = (G_\Omega f, -\Delta \varphi)_\Omega = (\nabla^\perp G_\Omega f, \nabla^\perp \varphi)_\Omega \\ &= (\mathbf{K}_\Omega f, \nabla^\perp \varphi)_\Omega = (\text{curl } \mathbf{K}_\Omega f, \varphi)_\Omega - \int_{\partial\Omega} \varphi (\mathbf{K}_\Omega f)^\perp \cdot \mathbf{n}, \end{aligned}$$

and the last term on the right hand side is zero. We integrated by parts in the last equality, which is legitimate since $\mathbf{K}_\Omega f \in \mathbf{W}^{1,p}(\Omega)$. \square

2.5. The spaces $\mathbf{V}^{1,p}$. The evolution of our solution to the Euler system (P) will take place in the space of \mathbf{L}^p tangential vector-fields with L^p vorticity. That is, we define, for $1 < p < \infty$,

$$(2.20) \quad \mathbf{V}^{1,p}(\Omega) := \{\mathbf{v} \in \mathbf{L}^p_\tau(\Omega) : \operatorname{curl} \mathbf{v} \in L^p(\Omega)\}, \quad \|\mathbf{v}\|_{\mathbf{V}^{1,p}(\Omega)} := \left(\|\mathbf{v}\|_{\mathbf{L}^p_\tau(\Omega)}^p + \|\operatorname{curl} \mathbf{v}\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}.$$

As a consequence of Lemma 2.3 and Theorem 2.2, when $1 < p \leq 2$, we have the continuous embedding $\mathbf{V}^{1,p}(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$:

$$(2.21) \quad \|\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C_{p,\Omega} \|\mathbf{v}\|_{\mathbf{V}^{1,p}(\Omega)} \quad \forall \mathbf{v} \in \mathbf{V}^{1,p}(\Omega), \quad 1 < p \leq 2.$$

The embedding discussed above allows for an improvement of the boundary regularity of functions in $\mathbf{V}^{1,p}(\Omega)$, by further applying Gagliardo's trace theorem to the components of \mathbf{v} . More precisely,

$$(2.22) \quad \mathbf{v}|_{\partial\Omega} \in \mathbf{W}^{1-\frac{1}{p},p}(\partial\Omega), \quad \forall 1 < p \leq 2.$$

In view of (2.22), we can therefore make sense of $\gamma_{\mathbf{n}} \mathbf{v}$ as $\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega}$ whenever $\mathbf{v} \in \mathbf{V}^{1,p}(\Omega)$.

3. ELLIPTIC REGULARITY IN A RECTANGLE

We assume throughout this section that $\Omega = [0, L_1] \times [0, L_2]$. Hereafter, we develop an appropriate substitute of Theorem 2.2 in the range $2 < p \leq \infty$, with spaces of functions with bounded mean oscillation replacing L^∞ .

3.1. Local bmo spaces. We denote by $\operatorname{bmo}(\mathbb{R}^2)$ the strict subspace of $\operatorname{BMO}(\mathbb{R}^2)$ normed by

$$(3.1) \quad \|f\|_{\operatorname{bmo}(\mathbb{R}^2)} = \sup_{|Q| < 1} \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx + \sup_{|Q| \geq 1} \frac{1}{|Q|} \int_Q |f(x)| dx$$

where the suprema above are taken over squares $Q \subset \mathbb{R}^2$, $|Q|$ denotes the area of Q , and f_Q denotes the average of f on the cube Q . See for example [6] for more details.

Let $D \subset \mathbb{R}^2$ be a domain. For a function $f : \overline{D} \rightarrow \mathbb{R}$, let \bar{f} be its extension to zero outside \overline{D} , i.e. $\bar{f} := f \mathbf{1}_{\overline{D}}$. We define the Banach spaces

$$\operatorname{bmo}_z(D) = \{f : \overline{D} \rightarrow \mathbb{R} : \bar{f} \in \operatorname{bmo}(\mathbb{R}^2)\}, \quad \|f\|_{\operatorname{bmo}_z(D)} := \|\bar{f}\|_{\operatorname{bmo}(\mathbb{R}^2)},$$

i.e. the space of functions on \overline{D} whose trivial extension is in $\operatorname{bmo}(\mathbb{R}^2)$. We also introduce the larger space and

$$\operatorname{bmo}_r(D) = \{f : D \rightarrow \mathbb{R} : \exists F \in \operatorname{bmo}(\mathbb{R}^2) \text{ with } F|_D = f\}, \quad \|f\|_{\operatorname{bmo}_r(D)} := \inf_{\substack{F \in \operatorname{bmo}(\mathbb{R}^2) \\ F|_D = f}} \|F\|_{\operatorname{bmo}(\mathbb{R}^2)},$$

i.e. the space of functions on D which are restrictions to D of functions in $\operatorname{bmo}(\mathbb{R}^2)$. The continuous embeddings $L^\infty(D) \hookrightarrow \operatorname{bmo}_z(D) \hookrightarrow \operatorname{bmo}_r(D)$ are immediate to verify; this, together with John-Nirenberg's inequality

$$(3.2) \quad \|f\|_{L^p(D)} \leq C_D p \|f\|_{\operatorname{bmo}_r(D)},$$

where the constant C_D is only dependent on $\operatorname{diam}(D)$, hints at the relevance of $\operatorname{bmo}_z(D)$ and $\operatorname{bmo}_r(D)$ as a substitute for $L^\infty(D)$. For $\star \in \{z, r\}$, we use the notation

$$W^{2,\operatorname{bmo}_\star}(D) = \{f \in \operatorname{bmo}_\star(D) : D^2 f \in \operatorname{bmo}_\star(D)^{2 \times 2}\}, \quad \|f\|_{W^{2,\operatorname{bmo}_\star}(D)} := \|f\|_{\operatorname{bmo}_\star(D)} + \|D^2 f\|_{\operatorname{bmo}_\star(D)^{2 \times 2}}.$$

The next theorem, which we quote from [6], tells us that the solution to the Dirichlet problem $G_D f$ is in $W^{2,\operatorname{bmo}_z}(D)$ whenever $f \in \operatorname{bmo}_z(D)$.

Theorem 3.1. *Let D be either a bounded domain of class \mathcal{C}^2 or the halfspace \mathbb{R}_+^2 . Then there exists a constant $C_D > 0$, depending only on D , such that*

$$(3.3) \quad G_D : \text{bmo}_z(D) \rightarrow W^{2,\text{bmo}_z}(D), \quad \|G_D f\|_{W^{2,\text{bmo}_z}(D)} \leq C_D \|f\|_{\text{bmo}_z(D)}.$$

3.2. Theorem 3.1 in a rectangle. We prove the following proposition, which is a (partial) extension of Theorem 3.1 to the (nonsmooth, convex) domain $\Omega = [0, L_1] \times [0, L_2]$.

Proposition 3.1. *Let $\Omega = [0, L_1] \times [0, L_2]$. There exists $C = C(L_1, L_2)$ such that*

$$(3.4) \quad \|G_\Omega f\|_{W^{2,\text{bmo}_r}(\Omega)} \leq C \|f\|_{\text{bmo}_z(\Omega)}.$$

Proof. In this proof, the almost inequality sign is hiding a positive constant, possibly varying from line to line and depending on Ω only. We also refer to Remark 6.3 for notation.

Denote by $F := G_\Omega f$. We preliminarily observe that, by Grisvard's classical $L^p - W^{2,p}$ estimate for domains with corners of aperture less than or equal to $\frac{\pi}{2}$ (which we recall in Section 6, see (6.7)) applied for $p = 3$, and John-Nirenberg's inequality (3.2)

$$\|F\|_{W^{2,3}(\Omega)} \lesssim \|f\|_{L^3(\Omega)} \lesssim \|f\|_{\text{bmo}_z(\Omega)}.$$

By the Sobolev embedding $W^{2,3}(\Omega) \subset \mathcal{C}^{1,\frac{1}{3}}(\overline{\Omega})$, we have in particular that

$$(3.5) \quad \nabla F \in \mathcal{C}^{0,\frac{1}{3}}(\overline{\Omega})^2, \quad \|F\|_{L^\infty(\Omega)} + \|\nabla F\|_{L^\infty(\Omega)} \lesssim \|f\|_{\text{bmo}_z(\Omega)}.$$

The core of the argument begins now. Let $\mathcal{C} := \{\Omega_0 \cup \{B_i : i = 1, \dots, 4\}\}$ be an open cover of $\overline{\Omega}$ as follows: each B_i is an open ball centered at the corner S_i , of radius $\rho > 0$ chosen small enough so that $B_i \cap B_j = \emptyset$ when $1 \leq i < j \leq 4$; $\Omega_0 \subset \Omega$ is an open set with smooth boundary and $\text{dist}(\overline{\Omega_0}, \{S_1, \dots, S_4\}) > \frac{\rho}{2}$. For $i = 1, \dots, 4$, set $\Omega_i = B_i \cap \Omega$. Let $\{\mu_0, \mu_1, \dots, \mu_4\}$ be a partition of unity subordinated to the cover \mathcal{C} , and write $F_i = F\mu_i$. Then F_i solves the Dirichlet problem

$$(3.6) \quad \begin{cases} \Delta F_i = f_i & \text{on } \Omega_i, \\ F_i = 0 & \text{on } \partial\Omega_i, \end{cases} \quad f_i = \mu_i f + \nabla F \cdot \nabla \mu_i + F \Delta \mu_i, \quad i = 0, \dots, 4.$$

We gather from (3.5) that

$$(3.7) \quad \|f_i\|_{\text{bmo}_z(\Omega)} \lesssim \|\mu_i\|_{C^2(\Omega)} (\|f\|_{\text{bmo}_z(\Omega)} + \|\nabla F\|_{L^\infty(\Omega)} + \|F\|_{L^\infty(\Omega)}) \lesssim \|f\|_{\text{bmo}_z(\Omega)}.$$

Since Ω_0 is a smooth domain, we apply Theorem 3.1 with $D = \Omega_0$, and get

$$(3.8) \quad \|F_0\|_{W^{2,\text{bmo}_r}(\Omega)} = \|F_0\|_{W^{2,\text{bmo}_z}(\Omega_0)} \lesssim \|f_0\|_{\text{bmo}_z(\Omega_0)} \lesssim \|f\|_{\text{bmo}_z(\Omega)}.$$

We now deal with the cases $i \geq 1$ and show that

$$(3.9) \quad \|F_i\|_{W^{2,\text{bmo}_r}(\Omega)} \lesssim \|f_i\|_{\text{bmo}_z(\Omega)}, \quad i = 1, \dots, 4$$

which will be enough to obtain (3.4). The four corners are treated in the exact same way, so we fix $i = 1$, and $S_1 = (0, 0)$. Note that F_1 solves the Dirichlet problem

$$\begin{cases} \Delta F_1 = f_1 & \text{on } \Gamma : (0, \infty) \times (0, \infty), \\ F_1(x_1, 0) = 0 & x_1 \geq 0, \\ F_1(0, x_2) = 0 & x_2 \geq 0. \end{cases}$$

Consider the odd reflections of F_1 and f_1 , defined on \mathbb{R}_+^2 by:

$$w(x_1, x_2) := \begin{cases} F_1(x_1, x_2) & x_1 \geq 0, x_2 > 0 \\ -F_1(-x_1, x_2) & x_1 < 0, x_2 > 0 \end{cases}, \quad h(x_1, x_2) := \begin{cases} f_1(x_1, x_2) & x_1 > 0, x_2 > 0 \\ -f_1(-x_1, x_2) & x_1 < 0, x_2 > 0 \end{cases}.$$

One sees that w is the solution to the Dirichlet problem on \mathbb{R}_+^2 with data h and zero boundary condition, i.e. $w = G_{\mathbb{R}_+^2} h$. We note that $\|h\|_{\text{bmo}_z(\mathbb{R}_+^2)} \leq 2\|f_1\|_{\text{bmo}_z(\Gamma)}$; indeed, this is the same as $\|\bar{h}\|_{\text{bmo}(\mathbb{R}^2)} \leq 2\|\bar{f}_1\|_{\text{bmo}(\mathbb{R}^2)}$; recall that the bar denotes extension by zero to \mathbb{R}^2 . This is clear, since $\bar{h}(x_1, x_2) = \bar{f}_1(x_1, x_2) - \bar{f}_1(-x_1, x_2)$. Therefore, we can apply Theorem 3.1 in the case of $D = \mathbb{R}_+^2$, and deduce that

$$(3.10) \quad \|w\|_{W^{2,\text{bmo}_z}(\mathbb{R}_+^2)} \lesssim \|h\|_{\text{bmo}_z(\mathbb{R}_+^2)} \lesssim \|f_1\|_{\text{bmo}_z(\Gamma)} = \|f_1\|_{\text{bmo}_z(\Omega)}.$$

Since w restricted to Γ coincides with F_1 , and F_1 is supported in Γ , we have

$$\|F_1\|_{W^{2,\text{bmo}_r}(\Omega)} = \|F_1\|_{W^{2,\text{bmo}_r}(\Gamma)} \leq \|w\|_{W^{2,\text{bmo}_z}(\mathbb{R}_+^2)},$$

and thus (3.9) follows from (3.10). The proof of Proposition 3.1 is complete. \square

3.3. The space $\mathbf{V}^{1,\infty}(\Omega)$. In view of the elliptic regularity result (3.4), for $\Omega = [0, L_1] \times [0, L_2]$ it makes sense to extend the scale of spaces $\mathbf{V}^{1,p}(\Omega)$ to the endpoint $p = \infty$ by setting

$$(3.11) \quad \mathbf{V}^{1,\infty}(\Omega) := \{\mathbf{v} \in \mathbf{L}_\tau^2(\Omega) \cap \mathbf{L}^\infty(\Omega) : \text{curl } \mathbf{v} \in L^\infty(\Omega)\}, \quad \|\mathbf{v}\|_{\mathbf{V}^{1,\infty}(\Omega)} := \|\mathbf{v}\|_{\mathbf{L}^\infty(\Omega)} + \|\text{curl } \mathbf{v}\|_{L^\infty(\Omega)}.$$

As a consequence of Proposition 3.1, we have the continuous embedding

$$(3.12) \quad \mathbf{V}^{1,\infty}(\Omega) \hookrightarrow \mathbf{W}^{1,\text{bmo}_z}(\Omega) = \{\mathbf{v} \in \text{bmo}_z(\Omega)^2 : \nabla \mathbf{v} \in \text{bmo}_z(\Omega)^{2 \times 2}\}$$

and in turn, John-Nirenberg inequality gives in particular that

$$(3.13) \quad \|\nabla \mathbf{v}\|_{L^p(\Omega)^{2 \times 2}} \leq C p \|\mathbf{v}\|_{\mathbf{V}^{1,\infty}(\Omega)} \quad \forall p \in [1, \infty).$$

A further remark is that standard interpolation theory [30] yields the embedding $\mathbf{V}^{1,p}(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$ for all $1 < p < \infty$ as well when Ω is a rectangle.

4. THE TWO-DIMENSIONAL EULER EQUATIONS IN A BOUNDED CONVEX DOMAIN

In this section, we formulate problem (P) in a suitable weak sense and construct a weak solution.

4.1. Weak formulation of (P) in bounded convex domains. Our first task is to provide a weak formulation of (P) when Ω is a bounded convex domain, which we continue to assume throughout this section. For convenience, we say that a triple of exponents (r_1, r_2, r_3) is a Hölder triplet if

$$1 \leq r_1, r_2, r_3 \leq \infty, \quad \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1.$$

4.1.1. *The trilinear form.* We define the trilinear form

$$b_\Omega(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} = ((\mathbf{u} \cdot \nabla) \mathbf{v}, \mathbf{w})_\Omega, \quad \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{D}(\bar{\Omega}; \mathbb{R}^2).$$

Lemma 4.1. *Let (r_1, r_2, r_3) be a Hölder triplet. Then the form b_Ω extends to a trilinear continuous form*

$$b_\Omega : \mathbf{L}_\tau^{r_1}(\Omega) \times \mathbf{W}^{1, r_2}(\Omega) \times \mathbf{L}^{r_3}(\Omega);$$

moreover, we have

$$(4.1) \quad b_\Omega(\mathbf{u}, \mathbf{v}, \mathbf{w}) + b_\Omega(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 0, \quad \forall \mathbf{u} \in \mathbf{L}_\tau^{r_1}(\Omega), \quad \mathbf{v}, \mathbf{w} \in \mathbf{W}^{1, r_2}(\Omega) \cap \mathbf{L}^{r_3}(\Omega).$$

Proof. Note that, as a mere consequence of Hölder's inequality

$$(4.2) \quad |b_\Omega(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \|\mathbf{u}\|_{\mathbf{L}^1(\Omega)} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{w}\|_{\mathbf{L}^3(\Omega)}, \quad \forall \mathbf{u} \in \mathcal{V}, \mathbf{v}, \mathbf{w} \in \mathcal{D}(\bar{\Omega}; \mathbb{R}^2).$$

Thus the first claim follows by density of \mathcal{V} in $\mathbf{L}^1(\Omega)$ and of $\mathcal{D}(\bar{\Omega}; \mathbb{R}^2)$ in $\mathbf{W}^{1, r_2}(\Omega) \cap \mathbf{L}^{r_3}(\Omega)$. To show (4.1), we employ Stokes' formula for $\mathbf{u} \in \mathcal{V}$ and $\mathbf{v} \in \mathcal{D}(\bar{\Omega}; \mathbb{R}^2)$ to find

$$2b_\Omega(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 2 \sum_{j=1}^2 (\mathbf{u}, \nabla(v_j)^2)_\Omega = - \sum_{j=1}^2 (\operatorname{div} \mathbf{u}, (v_j)^2)_\Omega + \sum_{j=1}^2 \int_{\partial\Omega} (v_j)^2 \mathbf{u} \cdot \mathbf{n} = 0,$$

and (4.1) then follows by trilinearity and density. \square

Remark 4.1. This remark, together with Remark 4.3 below, motivates why we are mostly interested in the range $p \in [\frac{4}{3}, \infty)$. Define

$$(4.3) \quad s(p) = \begin{cases} \frac{2p}{4-p}, & \frac{4}{3} \leq p < 2, \\ \text{any } s \in [1, 2) & p = 2 \\ p & p > 2 \end{cases}, \quad z(p) = (s(p))'.$$

Assume $\mathbf{u} \in \mathbf{V}^{1, p}(\Omega)$. If $p \neq 2$, taking $r_1 = p^*, r_2 = p, r_3 = z(p)$ in (4.2), and exploiting the Sobolev embedding $\mathbf{V}^{1, p}(\Omega) \subset \mathbf{L}^{p^*}(\Omega)$ yields

$$(4.4) \quad \|\mathbf{v} \mapsto b_\Omega(\mathbf{u}, \mathbf{u}, \mathbf{v})\|_{\mathcal{L}(\mathbf{L}^{z(p)}(\Omega), \mathbb{R})} \leq C_\Omega \|\mathbf{u}\|_{\mathbf{V}^{1, p}(\Omega)}^2.$$

By the Riesz representation theorem, moreover, it follows that $(\mathbf{u} \cdot \nabla) \mathbf{u} \in \mathbf{L}^{s(p)}(\Omega)$ whenever $p \in (\frac{4}{3}, \infty)$. A similar argument gives the same result for any choice of $s(p) \in [1, 2)$ when $p = 2$.

Let $T > 0$ be fixed. We will also make use of the functional

$$(4.5) \quad B_\Omega(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_0^T b_\Omega(\mathbf{u}(t), \mathbf{v}(t), \mathbf{w}(t)) dt$$

initially defined on $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{C}^\infty([0, T] \times \bar{\Omega})$. In consequence of Lemma 4.1 and Hölder's inequality, whenever $\{s_i\}_{i=1}^3, \{r_i\}_{i=1}^3$ are Hölder triplets, the estimates

$$(4.6) \quad |B_\Omega(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \|\mathbf{u}\|_{L^{s_1}(0, T; \mathbf{L}_\tau^{r_1}(\Omega))} \|\mathbf{v}\|_{L^{s_2}(0, T; \mathbf{W}^{1, r_2}(\Omega))} \|\mathbf{w}\|_{L^{s_3}(0, T; \mathbf{L}^{r_3}(\Omega))}$$

hold for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{C}^\infty([0, T] \times \bar{\Omega})$ and thus B_Ω extends to a trilinear continuous form on the triple of spaces appearing in the right-hand side of (4.6). Analogously,

$$(4.7) \quad B_\Omega(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -B_\Omega(\mathbf{u}, \mathbf{w}, \mathbf{v}) \quad \forall \mathbf{u} \in L^{s_1}(0, T; \mathbf{L}_\tau^{r_1}(\Omega)), \mathbf{v}, \mathbf{w} \in L^{s_2}(0, T; \mathbf{W}^{1, r_2}(\Omega)) \cap L^{s_3}(0, T; \mathbf{L}^{r_3}(\Omega)).$$

4.1.2. *Weak solutions.* Let $p \in [\frac{4}{3}, \infty)$, $\mathbf{u}_0 \in \mathbf{V}^{1,p}(\Omega)$, $\mathbf{f} \in L^p(0, T; \mathbf{W}^{1,p}(\Omega))$ be given. A function

$$\mathbf{u} : \Omega \times (0, T) \rightarrow \mathbb{R}^2$$

is a *weak solution* to problem (P) if

$$(4.8) \quad \mathbf{u} \in L^\infty(0, T; \mathbf{V}^{1,p}(\Omega));$$

$$(4.9) \quad \mathbf{u} \in \mathcal{C}([0, T]; \mathbf{L}_\tau^p(\Omega)) \quad \text{and} \quad \mathbf{u}(0) = \mathbf{u}_0,$$

$$(4.10) \quad - \int_0^T (\mathbf{u}(t), \mathbf{v} \psi'(t))_\Omega dt + B_\Omega(\mathbf{u}, \mathbf{u}, \mathbf{v} \otimes \psi) = \int_0^T (\mathbf{f}(t), \mathbf{v} \psi(t))_\Omega,$$

for all $\mathbf{v} \in \mathcal{V}$, and $\psi \in \mathcal{D}(0, T)$.

Remark 4.2. Note that \mathcal{V} is certainly not dense in $\mathbf{V}^{1,p}(\Omega)$. In fact, one can see that the closure of \mathcal{V} in $\mathbf{V}^{1,p}(\Omega)$ is $\mathbf{V}^{1,p}(\Omega) \cap \mathbf{W}_0^{1,p}(\Omega)$. In a smooth domain Ω , one can obtain $\mathbf{V}^{1,p}(\Omega)$ as the closure of

$$\mathcal{W} := \{\mathbf{v} \in \mathcal{C}^\infty(\bar{\Omega}) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0\} = \mathcal{C}^\infty(\bar{\Omega}) \cap \mathbf{V}^{1,p}(\Omega).$$

This is not known to be true (and may very well be false) in a general convex domain. Ultimately, we are interested in testing the equation (4.10) with functions of $\mathbf{V}^{1,p}(\Omega)$, thus the choice of \mathcal{W} as a test function space over \mathcal{V} does not give any advantage. We explain how to circumvent this difficulty in Remark 4.3.

Remark 4.3. Assume that $p \geq \frac{4}{3}$ is such that the embedding $\mathbf{V}^{1,p}(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$ holds, and \mathbf{u} is a weak solution of (P). Note that each of the three functionals

$$\mathbf{v} \mapsto \int_0^T (\mathbf{u}, \mathbf{v} \psi')_\Omega, \quad \mathbf{v} \mapsto \int_0^T (\mathbf{f}, \mathbf{v} \psi)_\Omega, \quad \mathbf{v} \mapsto B_\Omega(\mathbf{u}, \mathbf{u}, \mathbf{v} \otimes \psi),$$

is linear and continuous on $\mathbf{L}_\tau^{z(p)}(\Omega)$: the only nontrivial verification is that, thanks to (4.4) and the embedding

$$\begin{aligned} |B_\Omega(\mathbf{u}, \mathbf{u}, \mathbf{v} \otimes \psi)| &\leq C_\Omega \|\mathbf{u}\|_{L^\infty(0, T; \mathbf{L}_\tau^p(\Omega))} \|\mathbf{u}\|_{L^p(0, T; \mathbf{W}^{1,p}(\Omega))} \|\mathbf{v} \otimes \psi\|_{L^{p'}(0, T; \mathbf{L}_\tau^{z(p)}(\Omega))} \\ &\leq C \|\mathbf{u}\|_{L^p(0, T; \mathbf{V}^{1,p}(\Omega))}^2 \|\psi\|_{L^{p'}(0, T)} \|\mathbf{v}\|_{\mathbf{L}_\tau^{z(p)}(\Omega)}. \end{aligned}$$

This shows that (4.10) makes sense for \mathbf{u} as in (4.8). By density of \mathcal{V} in $\mathbf{L}_\tau^{z(p)}(\Omega)$, the equality (4.10) actually holds for all $\mathbf{v} \in \mathbf{L}_\tau^{z(p)}(\Omega)$. Further observe that $p \geq \frac{3}{2}$ ensures the continuity of the embedding $\mathbf{W}^{1,p}(\Omega) \hookrightarrow \mathbf{L}^{z(p)}(\Omega)$. Hence, when $p \geq \frac{3}{2}$, it is legitimate to take $\mathbf{v} \in \mathbf{V}^{1,p}(\Omega)$ as a test function in (4.10).

We summarize the construction of a weak solution to the Euler system (P) in the proposition below.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain and $p \in [\frac{4}{3}, 2]$. Then Problem (P) has a weak solution $\mathbf{u} = \mathbf{u}(x, t) \in L^\infty(0, T; \mathbf{V}^{1,p}(\Omega))$ in the sense of (4.8)-(4.10), which possesses the further regularity*

$$(4.11) \quad \mathbf{u} \in \mathcal{C}([0, T]; \mathbf{L}_\tau^2(\Omega)), \quad \mathbf{u}(t) \in \mathbf{V}^{1,p}(\Omega) \quad \forall t \in [0, T],$$

$$(4.12) \quad \operatorname{curl} \mathbf{u} \in \mathcal{C}([0, T]; \mathbf{w} - L^p(\Omega)),$$

$$(4.13) \quad \partial_t \mathbf{u} \in L^p(0, T; \mathbf{L}^{s(p)}(\Omega)).$$

Remark 4.4. Proposition 4.1 is a slight improvement of [31, Proposition 7.1]. The proof follows the same general scheme of construction of a weak solution to (P) on Ω as a limit of smooth solutions on a sequence of approximating domains. However, in [31], no continuity-in-time results analogous to (4.11)-(4.12) are given.

4.2. Proof of Proposition 4.1. We assume throughout the proof that Ω is a bounded convex domain, $\frac{4}{3} \leq p \leq 2$, and

$$(4.14) \quad \mathbf{u}_0 \in \mathbf{V}^{1,p}(\Omega), \quad \mathbf{f} \in L^p(0, T; \mathbf{W}^{1,p}(\Omega)), \quad R := \left(\|\mathbf{u}_0\|_{\mathbf{V}^{1,p}(\Omega)}^p + \|\mathbf{f}\|_{L^p(0, T; \mathbf{W}^{1,p}(\Omega))}^p \right)^{\frac{1}{p}}.$$

The symbol \mathcal{Q} will stand for a positive increasing function of its argument, depending only on p and Ω , which can be explicitly computed and is allowed to vary from line to line. Also, all the constants implied by the almost inequality signs (\lesssim) are allowed to depend on p and Ω without explicit mention. The proof will proceed through several steps.

4.2.1. Approximation of the domain and of the data. We approximate Ω by a sequence of smooth convex domains Ω_n as in (2.1). As discussed in Section 2, the implicit constants in the Sobolev embeddings for Ω_n , as well as the $\mathbf{L}^p(\Omega_n)$ norm of the Leray projector P_{Ω_n} are uniform in n and depend only on (the Lipschitz character of) Ω . Our use of almost-inequalities and of the function $\mathcal{Q}(\cdot)$ will reflect this uniformity.

We turn to the approximation of the data in (P). Let $\omega_0 = \text{curl } \mathbf{u}_0$. We set

$$(4.15) \quad \omega_0^{(n)}(x) = \left(\omega_0 \mathbf{1}_{\Omega_{n-1}} \right) * \phi_{\varepsilon_n}^2(x), \quad x \in \Omega,$$

$$(4.16) \quad \mathbf{f}^{(n)}(x, t) = \left((\mathbf{f} \mathbf{1}_{\Omega_{n-1} \times (0, T-\varepsilon_n)}) * (\phi_{\varepsilon_n}^2 \otimes \phi_{\varepsilon_n}^1) \right)(x, t), \quad x \in \Omega, t \in [0, T],$$

with $\varepsilon_n > 0$, $\varepsilon_n \rightarrow 0$ chosen small enough to have

$$(4.17) \quad \text{supp } \omega_0^{(n)} \Subset \Omega_n, \quad \text{supp } \mathbf{f}^{(n)} \Subset \Omega_n \times (0, T).$$

Note that

$$(4.18) \quad \omega_0^{(n)} \in \mathcal{D}(\Omega_n), \quad \|\omega_0^{(n)}\|_{L^p(\Omega)} \leq \|\omega_0\|_{L^p(\Omega)}, \quad \widetilde{\omega_0^{(n)}} \rightarrow \omega_0 \text{ in } L^p(\Omega),$$

$$(4.19) \quad \mathbf{f}^{(n)} \in \mathcal{D}(\Omega_n \times (0, T)), \quad \|\mathbf{f}^{(n)}\|_{L^p(0, T; \mathbf{W}^{1,p}(\Omega_n))} \leq \|\mathbf{f}\|_{L^p(0, T; \mathbf{W}^{1,p}(\Omega))},$$

$$(4.20) \quad \|\mathbf{f}^{(n)} - \mathbf{f}\|_{L^p(0, T; \mathbf{W}^{1,p}(\Omega_n))} \rightarrow 0, \quad n \rightarrow \infty.$$

Lemma 4.2. *Define*

$$\mathbf{u}_0^{(n)} : \Omega_n \rightarrow \mathbb{R}^2, \quad \mathbf{u}_0^{(n)} = \mathbf{K}_{\Omega_n} \omega_0^{(n)}.$$

We then have

$$(4.21) \quad \mathbf{u}_0^{(n)} \in \mathcal{C}^\infty(\overline{\Omega_n}, \mathbb{R}^2),$$

$$(4.22) \quad \nabla \cdot \mathbf{u}_0^{(n)} = 0 \text{ in } \Omega_n, \quad \mathbf{u}_0^{(n)} \cdot \mathbf{n}|_{\partial\Omega_n} = 0;$$

$$(4.23) \quad \|\mathbf{u}_0^{(n)}\|_{\mathbf{W}^{1,p}(\Omega_n)} \leq \mathcal{Q}(R).$$

Proof. Immediate from (4.18) and Theorem 2.2. □

4.2.2. *The approximating problems and a priori estimates.* We consider the following evolution problem:

$$(P_n) \quad \begin{cases} \partial_t \mathbf{u}(x, t) + (\mathbf{u} \cdot \nabla) \mathbf{u}(x, t) + \nabla \pi(x, t) = \mathbf{f}^{(n)}(x, t), & x \in \Omega_n, t \in (0, T) \\ \nabla \cdot \mathbf{u}(x, t) = 0 & x \in \Omega_n, t \in (0, T) \\ \mathbf{u} \cdot \mathbf{n}(x, t) = 0, & x \in \partial\Omega_n, t \in (0, T) \\ \mathbf{u}(x, 0) = \mathbf{u}_0^{(n)}(x) & x \in \Omega_n \end{cases}$$

Due to the smoothness (4.21) of the data $\mathbf{u}_0^{(n)}, \mathbf{f}^{(n)}$ and of the domain Ω_n , the classical result of Kato [22] (see also [2, 3]) yields the existence of a unique classical solution of (P_n) :

$$(\mathbf{u}^{(n)}, \pi^{(n)}) \in \mathcal{C}^\infty(\overline{\Omega_n} \times [0, T])^2 \times \mathcal{C}^\infty(\overline{\Omega_n} \times [0, T]).$$

Taking curl of the equation in (P_n) , we obtain the following equation for $\omega^{(n)}$:

$$(4.24) \quad \begin{cases} \partial_t \omega^{(n)}(x, t) + (\mathbf{u}^{(n)} \cdot \nabla) \omega^{(n)}(x, t) = g^{(n)}(x, t), & x \in \Omega_n, t \in (0, T), \\ \omega^{(n)}(x, 0) = \omega_0^{(n)}(x) & x \in \Omega_n, \end{cases}$$

where $g^{(n)} = \text{curl } \mathbf{f}^{(n)}$. We now derive *a priori* estimates on the smooth solutions $\mathbf{u}^{(n)}$ via the fundamental estimate on the vorticity $\omega^{(n)}$ in the lemma below.

Lemma 4.3. *Let R be as in (4.14). The following estimate holds:*

$$\|\omega^{(n)}\|_{L^\infty(0, T; L^p(\Omega_n))} \leq e^T R.$$

Proof. Multiplying (4.24) by $p\omega^{(n)}|\omega^{(n)}|^{p-2}$, and integrating on Ω_n , we obtain

$$\begin{aligned} \frac{d}{dt} \|\omega^{(n)}(t)\|_{L^p(\Omega_n)}^p &= p(g^{(n)}(t), \omega^{(n)}|\omega^{(n)}|^{p-2})_{\Omega_n} \leq \|g^{(n)}(t)\|_{L^p(\Omega_n)}^p + (p-1)\|\omega^{(n)}(t)\|_{L^p(\Omega_n)}^p \\ &\leq \|g^{(n)}(t)\|_{L^p(\Omega_n)}^p + (p-1)\|\omega^{(n)}(t)\|_{L^p(\Omega_n)}^p, \end{aligned}$$

so that the desired conclusion follows from the Gronwall lemma and (4.18)-(4.20). We used that

$$p((\mathbf{u}^{(n)} \cdot \nabla) \omega^{(n)}, \omega^{(n)}|\omega^{(n)}|^{p-2})_{\Omega_n} = (\mathbf{u}^{(n)}, \nabla |\omega^{(n)}|^p)_{\Omega_n} = 0$$

which stems from $\mathbf{u}^{(n)}$ being divergence free and with zero normal component on $\partial\Omega_n$. \square

Via an application of (2.16), Lemma 4.3 entails the following *a priori* estimate on the solutions $\mathbf{u}^{(n)}$:

$$(4.25) \quad \|\mathbf{u}^{(n)}\|_{L^\infty(0, T; \mathbf{V}^{1,p}(\Omega_n))} \leq \mathcal{Q}(R).$$

We then derive an equicontinuity result for the vorticity.

Lemma 4.4. *For each $\psi \in L^{p'}(\Omega)$, the sequence of functions*

$$t \in [0, T] \mapsto (\widetilde{\omega^{(n)}}(t), \psi)_\Omega,$$

is equicontinuous on $[0, T]$.

Proof. Let $\varepsilon > 0$ and $\psi \in L^{p'}(\Omega)$ be given; we have to show that

$$(4.26) \quad \exists \delta = \delta(R, \varepsilon, \psi), N = N(\varepsilon, \psi) : |t_2 - t_1| < \delta \implies \sup_{n \geq N} |(\widetilde{\omega^{(n)}}(t_2), \psi)_\Omega - (\widetilde{\omega^{(n)}}(t_1), \psi)_\Omega| < \varepsilon.$$

We first assume $\psi = \varphi \in \mathcal{D}(\Omega)$. Take $N = N(\varphi)$ large enough so that $\text{supp } \varphi \Subset \Omega_N$. Then, we multiply by φ the equation in (4.24) and integrate over Ω_n : observing that

$$\int_{\Omega_n} (\mathbf{u}^{(n)} \cdot \nabla \omega^{(n)} \varphi) + \int_{\Omega_n} (\mathbf{u}^{(n)} \cdot \nabla \varphi) \omega^{(n)} = - \int_{\Omega_n} (\text{div } \mathbf{u}^{(n)} \nabla \varphi) \omega^{(n)} - \int_{\partial \Omega_n} (\varphi \omega^{(n)}) \mathbf{u}^{(n)} \cdot \mathbf{n} = 0,$$

since $\mathbf{u}^{(n)}$ is divergence free and has zero normal component on $\partial \Omega_n$, and integrating on (t_1, t_2) , one finds

$$\begin{aligned} & \left| (\widetilde{\omega^{(n)}}(t_2), \varphi)_\Omega - \widetilde{\omega^{(n)}}(t_1), \varphi)_\Omega \right| = \left| (\omega^{(n)}(t_2), \psi)_{\Omega_n} - \omega^{(n)}(t_1), \psi)_{\Omega_n} \right| \\ & \leq \left| \int_{t_1}^{t_2} (\mathbf{u}^{(n)} \cdot \nabla \varphi, \omega^{(n)})_{\Omega_n} \right| + \left| \int_{t_1}^{t_2} (g^{(n)}, \varphi)_{\Omega_n} \right| \\ & \leq |t_1 - t_2| \|\mathbf{u}^{(n)}\|_{L^\infty(0, T; \mathbf{L}^{p'}(\Omega_n))} \|\nabla \varphi\|_{L^\infty(\Omega_n)} \|\omega^{(n)}\|_{L^\infty(0, T; L^p(\Omega_n))} + |t_1 - t_2|^{\frac{1}{p'}} \|g^{(n)}\|_{L^p(0, T; L^p(\Omega_n))} \|\varphi\|_{L^{p'}(\Omega_n)} \\ & \leq T^{\frac{1}{p}} |t_1 - t_2|^{p'} \left(\|\mathbf{u}^{(n)}\|_{L^\infty(0, T; \mathbf{V}^{1, p}(\Omega_n))} \|\nabla \varphi\|_{L^\infty(\Omega_n)} \|\omega^{(n)}\|_{L^\infty(0, T; L^p(\Omega_n))} + \|g^{(n)}\|_{L^p(0, T; L^p(\Omega_n))} \|\varphi\|_{L^{p'}(\Omega_n)} \right) \\ & \leq T^{\frac{1}{p}} |t_1 - t_2|^{p'} \mathcal{Q}(R) (\|\nabla \varphi\|_{L^\infty(\Omega_n)} + \|\varphi\|_{L^{p'}(\Omega_n)}). \end{aligned}$$

We used the Sobolev embedding $\mathbf{V}^{1, p}(\Omega_n) \hookrightarrow \mathbf{L}^{p'}(\Omega_n)$ and (4.25), as well as Lemma 4.3. Now, for $\psi \in L^{p'}(\Omega)$, take $\varphi \in \mathcal{D}(\Omega)$ with $\|\varphi - \psi\|_{L^{p'}(\Omega)} < \varepsilon/4 \|\omega^{(n)}\|_{L^\infty(0, T; L^p(\Omega))}$. Set

$$Q(\varepsilon, \psi, R) := T^{\frac{1}{p}} (\|\nabla \varphi\|_{L^\infty(\Omega)} + \|\varphi\|_{L^{p'}(\Omega)}).$$

Then for $n \geq N(\varphi) = N(\varepsilon, \psi)$,

$$\begin{aligned} \left| (\widetilde{\omega^{(n)}}(t_2) - \widetilde{\omega^{(n)}}(t_1), \psi)_\Omega \right| & \leq \left| (\widetilde{\omega^{(n)}}(t_2) - \widetilde{\omega^{(n)}}(t_1), \varphi)_\Omega \right| + \left| (\widetilde{\omega^{(n)}}(t_1) - \widetilde{\omega^{(n)}}(t_2), \varphi - \psi)_\Omega \right| \\ & \leq |t_1 - t_2|^{p'} Q(\varepsilon, \psi, R) + 2 \|\omega^{(n)}\|_{L^\infty(0, T; L^p(\Omega_n))} \|\varphi - \psi\|_{L^{p'}(\Omega)} < \varepsilon, \end{aligned}$$

if $|t_1 - t_2| \leq \delta := \left(\frac{\varepsilon}{2} Q(\varepsilon, \psi, R)\right)^{\frac{1}{p'}}$. Thus (4.26) is proven. \square

Denote $\Psi^{(n)} := G_{\Omega_n} \omega^{(n)}$. In view of Lemma 4.3, the elliptic regularity (2.11) gives

$$(4.27) \quad \Psi^n(t) \in W^{2, p}(\Omega_n) \cap W_0^{1, p}(\Omega_n) \quad \text{a.e. } t \in (0, T),$$

$$(4.28) \quad \|\Psi^{(n)}\|_{L^\infty(0, T; W^{2, p}(\Omega_n))} \lesssim \|\omega^{(n)}\|_{L^\infty(0, T; L^p(\Omega_n))} \leq \mathcal{Q}(R).$$

Note that the embedding $W^{2, p}(\Omega_n) \cap W_0^{1, p}(\Omega_n) \hookrightarrow W_0^{1, r}(\Omega_n)$ is continuous (with uniformity in n , see Section 2) whenever $1 \leq r < p^*$. Thus, we use the extension property (2.12) to derive from (4.27)-(4.27) that

$$(4.29) \quad \|\widetilde{\Psi^{(n)}}\|_{L^\infty(0, T; W_0^{1, r}(\Omega))} = \|\Psi^{(n)}\|_{L^\infty(0, T; W_0^{1, r}(\Omega_n))} \leq \mathcal{Q}(R), \quad 1 \leq r < p^*.$$

We will actually use (4.29) with $r = p$ and $r = 2$. Lastly, we derive a uniform estimate on the time-derivative of $\mathbf{u}^{(n)}$.

Lemma 4.5. *Let $s(p)$ be as in (4.3). We have the estimate*

$$(4.30) \quad \|\partial_t \widetilde{\Psi^{(n)}}\|_{L^p(0, T; W_0^{1, s(p)}(\Omega))} = \|\partial_t \mathbf{u}^{(n)}\|_{L^p(0, T; \mathbf{L}^{s(p)}(\Omega))} \leq \mathcal{Q}(R).$$

Proof. We set

$$\mathbf{w}^{(n)} = \nabla \Psi^{(n)} = (\mathbf{u}^{(n)})^\perp.$$

Observe that, since $\mathbf{w}^{(n)}$ is a gradient, $P_{\Omega_n}^\perp \partial_t \mathbf{w}^{(n)} = \partial_t P_{\Omega_n}^\perp \mathbf{w}^{(n)} = \partial_t \mathbf{w}^{(n)}$. At this point, note that, for any $\varphi \in \mathcal{D}(\Omega_n)$,

$$\begin{aligned} (\partial_t \mathbf{w}^{(n)}, \nabla \varphi)_{\Omega_n} &= \frac{d}{dt} (\mathbf{w}^{(n)}, \nabla \varphi)_{\Omega_n} = -\frac{d}{dt} (\Delta \Psi^{(n)}, \varphi)_{\Omega_n} = -(\partial_t \omega^{(n)}, \varphi)_{\Omega_n} \\ &= (\mathbf{u}^{(n)} \cdot \nabla \omega^{(n)}, \varphi)_{\Omega_n} - (g^{(n)}, \varphi)_{\Omega_n} \\ &= (-\omega^{(n)} \mathbf{u}^{(n)} + \nabla \mathbf{f}^{(n)}, \nabla \varphi)_{\Omega_n}. \end{aligned}$$

We integrated by parts the first term in the next to last line to arrive at the last line. Thus, we obtain the equation

$$(4.31) \quad \partial_t \mathbf{w}^{(n)} = -P_{\Omega_n}^\perp [-\omega^{(n)} \mathbf{u}^{(n)}] + \nabla \mathbf{f}^{(n)}, \quad \text{in } \Omega_n \times (0, T).$$

Observe that, by Sobolev embeddings and (4.28),

$$(4.32) \quad \|\mathbf{u}^{(n)}\|_{L^\infty(0,T;\mathbf{L}^r(\Omega_n))} \lesssim \|\Psi^{(n)}\|_{L^\infty(0,T;W^{2,p}(\Omega_n))} \lesssim \mathcal{Q}(R), \quad r = \begin{cases} p^*, & \frac{4}{3} \leq p < 2, \\ s(p)^*, & p = 2. \end{cases}$$

Theorem 2.1 ensures that $P_{\Omega_n}^\perp$ is a linear bounded operator on $\mathbf{L}^s(\Omega)$, with bound independent on n . This, together with Hölder's inequality and Sobolev embeddings gives

$$\begin{aligned} (4.33) \quad \|P_{\Omega_n}^\perp [\omega^{(n)} \mathbf{u}^{(n)}]\|_{L^\infty(0,T;\mathbf{L}^{s(p)}(\Omega_n))} &\lesssim \|\omega^{(n)} \mathbf{u}^{(n)}\|_{L^\infty(0,T;\mathbf{L}^{s(p)}(\Omega_n))} \\ &\leq \|\omega^{(n)}\|_{L^\infty(0,T;L^p(\Omega_n))} \|\mathbf{u}^{(n)}\|_{L^\infty(0,T;\mathbf{L}^r(\Omega_n))} \\ &\leq \mathcal{Q}(R). \end{aligned}$$

To obtain the last inequality we used Lemma 4.3 and (4.32). Since we also have

$$\|\nabla \mathbf{f}^{(n)}\|_{L^p(0,T;\mathbf{L}^s(\Omega_n))} \lesssim \|\nabla \mathbf{f}^{(n)}\|_{L^p(0,T;\mathbf{L}^p(\Omega_n))} \lesssim \|\mathbf{f}\|_{L^p(0,T;\mathbf{L}^p(\Omega))} \leq \mathcal{Q}(R),$$

we obtain that

$$\|\partial_t \mathbf{u}^{(n)}\|_{L^p((0,T),\mathbf{L}^{s(p)}(\Omega_n))} = \|\partial_t \mathbf{w}^{(n)}\|_{L^p((0,T),\mathbf{L}^{s(p)}(\Omega_n))} \leq \mathcal{Q}(R),$$

by comparison of (4.31) with (4.33) and with the last display. The lemma follows since $\widetilde{\partial_t \mathbf{u}^{(n)}} = \partial_t \widetilde{\mathbf{u}^{(n)}}$ in $\Omega \times [0, T]$ and the extension by zero to Ω preserves the L^p norms. \square

4.2.3. Conclusion of the proof. We will proceed in steps.

STEP 1. COMPACTNESS. In light of Lemmata 4.3 and 4.4, an application of Ascoli-Arzelà's Theorem yields that the sequence $\{\widetilde{\omega^{(n)}}\}$ is precompact in $\mathcal{C}([0, T]; w - L^p(\Omega))$, so that up to a subsequence

$$(4.34) \quad \widetilde{\omega^{(n)}} \rightarrow \omega \quad \text{in } \mathcal{C}([0, T]; w - L^p(\Omega)).$$

We derive some consequences from (4.34).

Lemma 4.6. *We have that*

$$(4.35) \quad \omega \in \mathcal{C}([0, T]; w - L^p(\Omega)) \cap L^\infty(0, T; L^p(\Omega)) \cap \mathcal{C}([0, T]; H^{-1}(\Omega)),$$

$$(4.36) \quad \omega(0) = \omega_0 \in L^p(\Omega).$$

$$(4.37) \quad \omega^{(n)}(t) \rightarrow \omega(t) \quad \text{in } H^{-1}(\Omega) \quad \forall t \in [0, T].$$

Proof. The first two inclusions in (4.35) have just been shown. Actually the first inclusion implies the stronger property

$$(4.38) \quad \omega(t) \in L^p(\Omega) \quad \forall t \in [0, T].$$

The third follows from the first and the continuity of the embedding $w - L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$ ¹. Then, (4.34) yields in particular that $\omega(0)$ is the $w - L^p(\Omega)$ -limit of $\{\omega^{(n)}(0)\}$; but, in view of (4.18), this implies $\omega(0) = \omega_0 \in L^p(\Omega)$, so that (4.36) follows. Finally, (4.37) follows from (4.34) and the continuity of the embedding $w - L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$. \square

Define

$$\Psi : \Omega \times [0, T] \rightarrow \mathbb{R}, \quad \Psi := G_\Omega \omega.$$

Then, an application of (2.2) and (4.35) yields

$$(4.39) \quad \Psi(t) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \quad \forall t \in [0, T], \quad \|\Psi\|_{L^\infty(0,T;W^{2,p}(\Omega))} \lesssim \|\omega\|_{L^\infty(0,T;L^p(\Omega))} \leq \mathcal{Q}(R).$$

Lemma 4.7 (Convergence of the stream functions). *We have that*

$$(4.40) \quad \widetilde{\Psi^{(n)}} \rightarrow \Psi := G_\Omega \omega \quad \text{in } L^2(0, T; w - H_0^1(\Omega)).$$

Proof. We need to show that

$$(\nabla \widetilde{\Psi^{(n)}}(\cdot), \nabla \varphi)_\Omega \rightarrow (\nabla \Psi(\cdot), \nabla \varphi)_\Omega \quad \text{in } L^2(0, T) \quad \forall \varphi \in H_0^1(\Omega), \|\varphi\|_{H_0^1(\Omega)} = 1.$$

Using Lemma 2.2, it follows from (4.37) that $\widetilde{\Psi^{(n)}}(t) \rightarrow \Psi(t)$ in $w - H_0^1(\Omega)$ for all t , that is

$$(\nabla \widetilde{\Psi^{(n)}}(t), \nabla \varphi)_\Omega \rightarrow (\nabla \Psi(t), \nabla \varphi)_\Omega \quad \forall t \in [0, T].$$

Then, the case $r = 2$ of (4.29) yields

$$|(\nabla \widetilde{\Psi^{(n)}}(t), \nabla \varphi)_\Omega| \leq \|\varphi\|_{H_0^1(\Omega)} \|\widetilde{\Psi^{(n)}}\|_{L^\infty(0,T;H_0^1(\Omega))} \lesssim \mathcal{Q}(R), \quad \forall t \in [0, T],$$

and a similar bound holds for Ψ , in view of (4.39). We conclude with the dominated convergence theorem. \square

We read from (4.29) with $r = 2$ and (4.30) that

$$(4.41) \quad \{\widetilde{\Psi^{(n)}}\} \text{ is bounded in } L^\infty(0, T; H_0^1(\Omega)), \quad \{\partial_t \widetilde{\Psi^{(n)}}\} \text{ is bounded in } L^p(0, T; L^{s(p)}(\Omega)).$$

Thus, via Aubin-Lions compactness lemma, the convergence (4.40) improves to the following two results:

$$(4.42) \quad \widetilde{\Psi^{(n)}} \rightarrow \Psi \quad \text{in } L^2(0, T; L^2(\Omega)),$$

$$(4.43) \quad \partial_t \Psi \in L^p(0, T; W^{1,s(p)}(\Omega)).$$

STEP 2. CONSTRUCTION OF THE SOLUTION TO (P). We now introduce

$$\mathbf{u} = \mathbf{u}(x, t) : \Omega \rightarrow \mathbb{R}^2, \quad \mathbf{u}(t) := \mathbf{K}_\Omega \omega(t), \quad t \in [0, T].$$

¹The continuity of the embedding $w - L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$ for $1 < p < \infty$ is a consequence of the compactness of $L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$. This in turn follows from the adjoint compact embedding $H_0^1(\Omega) \hookrightarrow L^{p'}(\Omega)$.

As a consequence of (2.16), Lemma 4.6, and (4.43) we have

$$(4.44) \quad \mathbf{u}(t) \in \mathbf{W}_\tau^{1,p}(\Omega) \quad \forall t \in [0, T],$$

$$(4.45) \quad \|\mathbf{u}\|_{L^\infty(0,T;\mathbf{W}_\tau^{1,p}(\Omega))} \lesssim \|\omega\|_{L^\infty(0,T;L^p(\Omega))} \leq \mathcal{Q}(R),$$

$$(4.46) \quad \mathbf{u} \in \mathcal{C}([0, T]; \mathbf{L}^2(\Omega)),$$

$$(4.47) \quad \partial_t \mathbf{u} \in L^p(0, T; \mathbf{L}^{s(p)}(\Omega)),$$

$$(4.48) \quad \mathbf{u}(0) = \mathbf{L}^2 - \lim_{t \rightarrow 0} \mathbf{u}(t) = \mathbf{K}_\Omega \omega_0 = \mathbf{u}_0$$

Finally, we preliminarily deduce from (4.40) that

$$(4.49) \quad \widetilde{\mathbf{u}^{(n)}} \rightarrow \mathbf{u} \quad \text{in } L^2(0, T; \mathbf{W} - \mathbf{L}^2(\Omega)).$$

Due to the convergences (4.42) and (4.34)

$$\begin{aligned} \|\widetilde{\mathbf{u}^{(n)}}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2 &= \int_0^T (\nabla^\perp \Psi^{(n)}(t), \nabla^\perp \Psi^{(n)}(t))_{\Omega_n} dt \\ &= - \int_0^T (\Psi^{(n)}(t), \Delta \Psi^{(n)}(t))_{\Omega_n} dt = \int_0^T (\Psi^{(n)}(t), \omega^{(n)}(t))_{\Omega} dt \\ &\rightarrow \int_0^T (\Psi(t), \omega(t))_{\Omega_n} dt = \|\mathbf{u}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}^2. \end{aligned}$$

Therefore (4.49) upgrades to

$$(4.50) \quad \widetilde{\mathbf{u}^{(n)}} \rightarrow \mathbf{u} \quad \text{in } L^2(0, T; \mathbf{L}^2(\Omega)).$$

STEP 3. PASSAGE TO THE LIMIT. Thanks to (4.45)-(4.48) we see that (4.8)-(4.9) and (4.13) hold true. To show that \mathbf{u} is a weak solution to (P) we are left with proving that the distributional equality (4.10) holds for any $\mathbf{v} \in \mathcal{V}$, $\psi \in \mathcal{D}(0, T)$.

Multiplying and integrating by parts in (P_n) leads to the equation

$$(4.51) \quad - \int_0^T (\mathbf{u}^{(n)}(t), \mathbf{v} \psi'(t))_{\Omega_n} dt + B_{\Omega_n}(\mathbf{u}^{(n)}, \mathbf{u}^{(n)}, \mathbf{v} \otimes \psi) = \int_0^T (\mathbf{f}(t), \mathbf{v} \psi(t))_{\Omega_n} dt.$$

Due to (4.50) and (4.20) respectively, it is immediate to see that the first term on the left hand side, and the right hand side of (4.51) converge to the homologous terms in (4.10). We now treat the nonlinear term. We have that, using (4.7), and subsequently (4.6) with $s_1 = r_1 = s_3 = r_3 = 2$, $s_2 = r_2 = \infty$,

$$\begin{aligned} &\left| B_{\Omega_n}(\mathbf{u}^{(n)}, \mathbf{u}^{(n)}, \mathbf{v} \psi) - B_\Omega((\mathbf{u}, \mathbf{u}, \mathbf{v} \psi)) \right| = \left| B_\Omega(\mathbf{u}^{(n)}, \mathbf{v} \psi, \mathbf{u}^{(n)}) - B_\Omega(\mathbf{u}, \mathbf{v} \psi, \mathbf{u}) \right| \\ &\leq \left| B_\Omega(\mathbf{u} - \widetilde{\mathbf{u}^{(n)}}, \mathbf{v} \psi, \mathbf{u}^{(n)}) \right| + \left| B_\Omega(\mathbf{u}, \mathbf{v} \psi, \mathbf{u} - \widetilde{\mathbf{u}^{(n)}}) \right| \\ &\lesssim \|\mathbf{u} - \widetilde{\mathbf{u}^{(n)}}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} (\|\widetilde{\mathbf{u}^{(n)}}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}) \|\psi\|_{L^\infty(0,T)} \|\nabla \mathbf{v}\|_{L^\infty(\Omega)}, \end{aligned}$$

so that (4.50) allows us to conclude that (4.10) holds. Note that $\{\widetilde{\mathbf{u}^{(n)}}\}$ is a bounded sequence in $L^2(0, T; \mathbf{L}^2(\Omega))$. Also note that for n sufficiently large, $\text{supp } \mathbf{v} \subset \Omega_n$, thus we could replace B_{Ω_n} by B_Ω in the second step. The proof of Proposition 2.2 is therefore complete.

5. MAIN RESULTS

Our first main result is an improvement of Proposition 4.1: we exploit the extra regularity (4.13) of the weak solution to Problem (P) given in Proposition 4.1 to recover the pressure π and show that the pair (\mathbf{u}, π) thus obtained satisfies (P) almost everywhere in $\Omega \times (0, T)$. We summarize the properties of the weak solutions obtained in the main results, which we term $(S)_p$ -weak solutions, in the following definition.

Definition 5.1. Let $\frac{4}{3} < p < \infty$, $\mathbf{u}_0 \in \mathbf{V}^{1,p}(\Omega)$, $\mathbf{f} \in L^p(0, T; \mathbf{W}^{1,p}(\Omega))$. Denote also $q = \max\{2, p\}$. We call $(S)_p$ -weak solution to the Euler system (P) with data \mathbf{u}_0, \mathbf{f} a pair

$$(\mathbf{u} : \Omega \times [0, T] \rightarrow \mathbb{R}^2, \pi : \Omega \times (0, T) \rightarrow \mathbb{R})$$

with $\mathbf{u}(0) = \mathbf{u}_0$, with the properties

$$(5.1) \quad \mathbf{u} \in \mathcal{C}([0, T]; \mathbf{L}_\tau^q(\Omega)) \cap L^\infty(0, T; \mathbf{V}^{1,p}(\Omega)), \quad \mathbf{u}(t) \in \mathbf{V}^{1,p}(\Omega) \quad \forall t \in [0, T],$$

$$(5.2) \quad \operatorname{curl} \mathbf{u} \in \mathcal{C}([0, T]; \mathbf{w} - L^p(\Omega))$$

$$(5.3) \quad \partial_t \mathbf{u}, \nabla \pi \in L^p(0, T; \mathbf{L}^{s(p)}(\Omega)),$$

$$(5.4) \quad \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla \pi = \mathbf{f} \quad \text{in } L^p(0, T; \mathbf{L}^{s(p)}(\Omega)) \text{ and a.e. in } \Omega \times (0, T).$$

Multiplying (5.4) by an appropriate test function and integrating on $\Omega \times (0, T)$, it is easy to see that an $(S)_p$ -weak solution to (P) is a weak solution in the sense of Subsection 4.1.2.

Theorem 5.1. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain and $p \in (\frac{4}{3}, 2]$. Given

$$\mathbf{u}_0 \in \mathbf{V}^{1,p}(\Omega), \quad \mathbf{f} \in L^p(0, T; \mathbf{W}^{1,p}(\Omega)),$$

there exists a pair $(\mathbf{u} : \Omega \times [0, T] \rightarrow \mathbb{R}^2, \pi : \Omega \times [0, T] \rightarrow \mathbb{R})$ which is an $(S)_p$ -weak solution to the Euler system (P) with data \mathbf{u}_0, \mathbf{f} .

Proof of Theorem 5.1. Assume $p \in (\frac{4}{3}, 2]$ and let \mathbf{u} be the solution to Problem (P) given by Proposition 4.1. Note that, in view of Remark 4.1 the functional $\mathbf{v} \mapsto B_\Omega(\mathbf{u}, \mathbf{u}, \mathbf{v})$ extends boundedly to $L^{p'}(0, T; \mathbf{L}_\tau^{z(p)}(\Omega))$, the function $(\mathbf{u} \cdot \nabla) \mathbf{u} \in L^\infty(0, T; \mathbf{L}^{s(p)}(\Omega))$, and we have the equality

$$(5.5) \quad B_\Omega(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \int_0^T ((\mathbf{u}(t) \cdot \nabla) \mathbf{u}(t), \mathbf{v}(t))_\Omega) dt, \quad \forall \mathbf{v} \in L^{p'}(0, T; \mathbf{L}_\tau^{z(p)}(\Omega)),$$

by density of $\mathcal{V} \otimes \mathcal{D}(0, T)$ in $L^{p'}(0, T; \mathbf{L}_\tau^{z(p)}(\Omega))$. Due to (4.13), we also have $\partial_t \mathbf{u} \in L^p(0, T; \mathbf{L}^{s(p)}(\Omega))$, and the equality

$$\int_0^T (\partial_t \mathbf{u}, \mathbf{v} \psi)_\Omega = - \int_0^T (\mathbf{u}, \mathbf{v} \psi')_\Omega = -B_\Omega(\mathbf{u}, \mathbf{u}, \mathbf{v} \otimes \psi) + \int_0^T (\mathbf{f}, \mathbf{v} \psi), \quad \forall \mathbf{v} \otimes \psi \in \mathcal{V} \otimes \mathcal{D}(0, T),$$

carries over by density to

$$(5.6) \quad \int_0^T \left(\partial_t \mathbf{u}(t) + (\mathbf{u}(t) \cdot \nabla) \mathbf{u}(t) - \mathbf{f}(t), \mathbf{v}(t) \right)_\Omega dt = 0, \quad \forall \mathbf{v} \in L^{p'}(0, T; \mathbf{L}_\tau^{z(p)}(\Omega)).$$

Now we define

$$(5.7) \quad \mathbf{w} : \Omega \times (0, T) \rightarrow \mathbb{R}^2, \quad \mathbf{w} := \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \mathbf{f};$$

we have $\mathbf{w} \in L^p(0, T; \mathbf{L}^{s(p)}(\Omega))$, and we read from (5.6) that \mathbf{w} belongs to the annihilator $X_{p'}$ of $\mathbf{L}^{p'}(0, T; \mathbf{L}^{z(p)}(\Omega))$ in $\mathbf{L}^{p'}(0, T; \mathbf{L}^{z(p)}(\Omega))$. Using (2.9), we see that $X_{p'}$ is the $L^p(0, T; \mathbf{L}^{s(p)}(\Omega))$ -closure of $(\mathbf{H}_\tau^{z(p)})^\perp \otimes \mathcal{D}(0, T)$, that is, $X_{p'} = L^p(0, T; \nabla W^{1,s(p)}(\Omega))$. Therefore, we proved that

$$(5.8) \quad \exists \pi \in L^p(0, T; W^{1,s(p)}(\Omega)) : \mathbf{w} = -\nabla \pi.$$

Summarizing (4.13), (5.5), (5.7), (5.8), Theorem 5.1 is proven. \square

We now come to the second main theorem, which deals with (P) on the rectangle $\Omega = [0, L_1] \times [0, L_2]$ in the range of exponents $2 < p \leq \infty$ not covered by Theorem 5.1. In particular, this theorem contains the uniqueness of weak solutions to (P) on a rectangle with bounded initial vorticity.

Theorem 5.2. *Let $\Omega = [0, L_1] \times [0, L_2]$ and $2 < p \leq \infty$, and*

$$\mathbf{u}_0 \in \mathbf{V}^{1,p}(\Omega), \quad \mathbf{f} \in L^p(0, T; \mathbf{W}^{1,p}(\Omega))$$

be given.

- (a) *If $p < \infty$, there exists a pair (\mathbf{u}, π) which is an $(S)_p$ -weak solution to the Euler system (P) with data \mathbf{u}_0, \mathbf{f} .*
- (b) *If $p = \infty$, there exists a unique (up to an additive constant in π) pair (\mathbf{u}, π) which is an $(S)_q$ -weak solution to the Euler system (P) with data \mathbf{u}_0, \mathbf{f} for all $2 < q < \infty$, with the additional regularity*

$$\mathbf{u} \in L^\infty(0, T; \mathbf{V}^{1,\infty}(\Omega)).$$

Proof of the existence in Theorem 5.2. Let for the moment $2 < p < \infty$ and \mathbf{u}_0, \mathbf{f} be given as in the theorem. The proof is carried out by an approximation procedure totally analogous to the one described in Subsections 4.2.1 and 4.2.2: with the same notation, the sequence $\{\tilde{\omega}^{(n)}\}$ is equicontinuous in $\mathcal{C}([0, T]; w - L^p(\Omega))$ and obeys the bound

$$(5.9) \quad \|\tilde{\omega}^{(n)}\|_{L^\infty(0, T; L^p(\Omega))} \leq e^T (\|\mathbf{u}_0\|_{\mathbf{V}^{1,p}(\Omega)} + \|\mathbf{f}\|_{L^p((0, T); \mathbf{W}^{1,p}(\Omega))})^{\frac{1}{p}} := e^T R_p.$$

The approximate solutions $\mathbf{u}^{(n)}$ are given by $\mathbf{u}^{(n)} = \mathbf{K}_{\Omega_n} \omega^{(n)}$. Thus we have that, uniformly in n , $\mathbf{u}^{(n)} \in L^\infty(0, T; \mathbf{V}^{1,p}(\Omega_n))$. In particular, the (uniform in n , see Section 2) embedding $\mathbf{V}^{1,p}(\Omega_n) \hookrightarrow \mathbf{W}^{1,p}(\Omega_n) \hookrightarrow \mathbf{L}^\infty(\Omega_n)$ then entails

$$(5.10) \quad \|\mathbf{u}^{(n)}\|_{L^\infty(0, T; \mathbf{L}^\infty(\Omega_n))} \leq \mathcal{Q}(R_p).$$

This allows to modify (4.33) appropriately and obtain the uniform estimate on the time-derivatives

$$(5.11) \quad \|\partial_t \widetilde{\mathbf{u}^{(n)}}\|_{L^p(0, T; \mathbf{L}^p(\Omega))} \leq \mathcal{Q}(R_p).$$

Letting ω be a limit point of $\{\tilde{\omega}^{(n)}\}$ in $\mathcal{C}([0, T]; w - L^p(\Omega))$ and defining the candidate solution $\mathbf{u} := \mathbf{K}_\Omega \omega$, an application of (6.7) also yields

$$\mathbf{u} \in L^\infty(0, T; \mathbf{V}^{1,p}(\Omega)) \hookrightarrow L^\infty(0, T; \mathbf{W}^{1,p}(\Omega)).$$

The same exact arguments as in Subsection 4.2.3 yield that \mathbf{u} is a weak solution to (P), satisfying in addition (5.1)-(5.3). Repeating the proof of Theorem 5.1 finally gives the remaining statements in the case $2 < p < \infty$.

The case $p = \infty$ can be dealt with as follows. One constructs an approximating sequence of problems as in Subsection 4.2.1, with data

$$(5.12) \quad \widetilde{\omega_0^{(n)}} \rightarrow \omega_0 \quad \text{in } L^\infty(\Omega), \quad \|\mathbf{f}^{(n)} - \mathbf{f}\|_{L^\infty(0,T;\mathbf{W}^{1,\infty}(\Omega_n))} \rightarrow 0, \quad n \rightarrow \infty.$$

Since the R_p in (5.9) are bounded uniformly in $2 < p < \infty$ by

$$R := R(\mathbf{u}_0, \mathbf{f}) := (\|\mathbf{u}_0\|_{\mathbf{V}^{1,\infty}(\Omega)} + \|\mathbf{f}\|_{L^\infty(0,T;\mathbf{W}^{1,\infty}(\Omega))})$$

one can pass to the limsup as $p \rightarrow \infty$, and obtain that

$$(5.13) \quad \|\widetilde{\omega^{(n)}}\|_{L^\infty(0,T;L^\infty(\Omega))} \leq e^T R.$$

Arguing similarly to what we did in Lemma 4.4 will give that the sequence $\{\widetilde{\omega^{(n)}}\}$ is precompact in $\mathcal{C}([0, T]; \mathbf{W}^* - L^\infty(\Omega))$.

One then chooses once again a limit point ω of $\{\widetilde{\omega^{(n)}}\}$ in $\mathcal{C}([0, T]; \mathbf{W}^* - L^\infty(\Omega))$ and defines the candidate solution $\mathbf{u} := \mathbf{K}_\Omega \omega$. An application of Proposition 3.1 then entails $\mathbf{u} \in L^\infty(0, T; \mathbf{V}^{1,\infty}(\Omega)) \hookrightarrow L^\infty(0, T; \mathbf{W}^{1,\mathbf{bmo}_z}(\Omega))$; in particular,

$$(5.14) \quad \|\mathbf{u}\|_{L^\infty(0,T;\mathbf{W}^{1,\mathbf{bmo}_z}(\Omega))} \leq CR(\mathbf{u}_0, \mathbf{f})e^T.$$

Here, one is not able to obtain a uniform $L^\infty(0, T; \mathbf{L}^\infty(\Omega))$ estimate on $\partial_t \widetilde{\mathbf{u}^{(n)}}$. We recall that $\partial_t \widetilde{\mathbf{u}^{(n)}}^\perp$ is given by equation (4.31). Despite $\omega^{(n)} \mathbf{u}^{(n)}$ being indeed uniformly bounded in $L^\infty(0, T; \mathbf{L}^\infty(\Omega_n))$, we do not have the (uniform) $\mathbf{L}^\infty(\Omega_n)$ boundedness of the Leray projector, nor do we have the substitute $P_{\Omega_n} \in \mathcal{L}(\mathbf{bmo}_z(\Omega_n))$ uniformly. However, the estimate (5.11) holds for all $p = q < \infty$. This is sufficient to carry on the arguments of Subsection 4.2.3 and conclude the proof of existence in the case $p = \infty$. \square

Proof of the uniqueness in Theorem 5.2. The proof is analogous to Yudovich's proof in the case of a smooth domain. With K we denote a positive constant, possibly varying from line to line and depending only on $(\mathbf{u}_0, \mathbf{f})$. Assume that there exist two solutions $(\mathbf{u}^1, \pi^1), (\mathbf{u}^2, \pi^2)$ corresponding to the same data $(\mathbf{u}_0, \mathbf{f})$. Preliminarily observe that, from (3.2), Proposition 3.1, and (5.14),

$$(5.15) \quad \begin{aligned} \|\nabla \mathbf{u}^j\|_{L^p(\Omega)} &\leq C_\Omega p \|\nabla \mathbf{u}^j\|_{\mathbf{bmo}_z(\Omega)^{2 \times 2}} \leq C_\Omega p \|\operatorname{curl} \mathbf{u}^j\|_{\mathbf{bmo}_z(\Omega)} \\ &\leq C_\Omega p \|\operatorname{curl} \mathbf{u}^j\|_{L^\infty(\Omega)} \leq C_\Omega p R(\mathbf{u}_0, \mathbf{f}) p := Kp \end{aligned}$$

for all $2 \leq p < \infty$. The difference $\mathbf{u} = \mathbf{u}^2 - \mathbf{u}^1$, $\pi = \pi^2 - \pi^1$ then satisfies the equation

$$(5.16) \quad \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}^1 + (\mathbf{u}^2 \cdot \nabla) \mathbf{u} + \nabla \pi = 0$$

almost everywhere in $\Omega \times (0, T)$. Denote by $Y(t) := \|\mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}^2$. We take the scalar product of (5.16) with \mathbf{u} and integrate on Ω . After (legitimately) integrating by parts, applying Hölder's inequality with an arbitrary $p > 2$ to be chosen later, and using interpolation one obtains the differential inequality

$$(5.17) \quad \frac{d}{dt} Y = -((\mathbf{u} \cdot \nabla) \mathbf{u}^1, \mathbf{u})_\Omega \leq K \|\nabla \mathbf{u}^1\|_{L^p(\Omega)} \|\mathbf{u}\|_{\mathbf{L}^{2p'}(\Omega)} \leq K \|\nabla \mathbf{u}^1\|_{L^p(\Omega)} \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{p}} \|\mathbf{u}\|_{\mathbf{L}^\infty(\Omega)}^{\frac{1}{p}} \leq Kp Y^{1-\frac{1}{p}};$$

in the last step, we used (5.14) to bound $\|\mathbf{u}\|_{\mathbf{L}^\infty(\Omega)}$, and (5.15). The bound $Y(t) \leq K$ for all $t \in [0, T]$ makes possible to choose a constant M large enough, depending only on $(\mathbf{u}_0, \mathbf{f})$ and

Ω , such that $p = p(t) = \log \frac{M}{Y(t)} > 2$ for all t . Choosing $p = p(t)$ in (5.17), and observing that with this choice $Y^{-\frac{1}{p}} \leq K$, the above differential inequality then turns into

$$\frac{d}{dt} Y(t) \leq K Y(t) \log \frac{M}{Y(t)}.$$

Let $\varepsilon > 0$ be given; integrating on (ε, t) , we obtain that

$$(5.18) \quad Y(t) \leq M(Y(\varepsilon)/M)^{e^{-K(t-\varepsilon)}}, \quad \forall t > \varepsilon.$$

Due to the continuity $\mathbf{u}^i \in \mathcal{C}([0, T]; \mathbf{L}_\tau^2(\Omega))$, the functional Y is continuous on $[0, T]$ and $Y(0) = 0$. Passing to the limit as $\varepsilon \rightarrow 0$ in (5.18) we obtain that $Y(t) \equiv 0$ on $[0, T]$. Thus $\mathbf{u}^1 = \mathbf{u}^2$, which forces $\nabla \pi^1 = \nabla \pi^2$ too. This completes the proof. \square

Remark 5.1. We further comment on the spatial regularity of the unique solution \mathbf{u} of Theorem 5.2 in the case $\mathbf{u}_0 \in \mathbf{V}^{1,\infty}(\Omega)$. Inequality (5.15) holding for each $2 \leq p < \infty$ implies, via standard arguments (see for example [1, Lemma 4.28]), that $\mathbf{u} = \mathbf{u}^j$ is a log-Lipschitz vector field on Ω , that is

$$\|\mathbf{u}\|_{\mathbf{LL}(\Omega)} := \|\mathbf{u}\|_{\mathbf{L}^\infty(\Omega)} + \sup_{\substack{x,y \in \Omega \\ 0 < |x-y| \leq 1}} \frac{|\mathbf{u}(x) - \mathbf{u}(y)|}{\psi_{LL}(|x-y|)} \lesssim K, \quad \psi_{LL}(s) := s \log(e + s^{-1}).$$

This is sufficient to ensure that the flow $(x_0, t) \mapsto x(t; x_0) := \Phi_t(x_0)$ generated by the Cauchy problem

$$\begin{cases} \dot{x}(t; x_0) = \mathbf{u}(x(t; x_0), t) & t > 0, \\ x(t; x_0) = x_0 \in \Omega \end{cases}$$

is uniquely defined. It is easy to check that the vorticity $\omega := \operatorname{curl} \mathbf{u}$ then satisfies

$$\omega(x, t) := \operatorname{curl} \mathbf{u}_0((\Phi_t)^{-1}(x)).$$

6. ADDITIONAL REMARKS

6.1. $(S)_p$ -weak solutions and turbulence. We show that, when $p \geq \frac{3}{2}$, $(S)_p$ -weak solutions obey conservation of kinetic energy. In this range, the embedding

$$L^\infty(0, T; \mathbf{V}^{1,p}(\Omega)) \hookrightarrow L^{p'}(0, T; \mathbf{L}_\tau^{z(p)}(\Omega))$$

is continuous (see also Remark 4.3). Therefore, we can multiply (5.4) by \mathbf{u} and integrate on $\Omega \times (0, T)$. The convective term $b_\Omega(\mathbf{u}, \mathbf{u}, \mathbf{u})$ vanishes identically, so that we get that the solution \mathbf{u} of Theorem 5.1 satisfies the energy equality

$$(6.1) \quad \frac{d}{dt} \|\mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)}^2 - (\mathbf{f}(t), \mathbf{u}(t))_\Omega = 0, \quad t \in (0, T).$$

In relation with (6.1), we recall that the issue of conservation of energy for the solutions of the Euler equations is an important issue related to turbulence, although not in the focus of this article. The relation of the conservation of energy for the Euler equations in relation with the Onsager conjecture [25] is well explained in e.g. the article by Schnirelman, [29] and its review in MathSciNet. In this direction Uriel Frisch just draw our attention to the most recent result of De Lellis and Székelyhidi Jr., [11] who proved the existence of solutions of the Euler equations which are Holder with exponent $< 1/10$ and which do not satisfy the conservation of energy; see also the earlier results [10]. More precisely [11] contains the

construction of solutions to the Euler equations with initial vorticity in $BV(\Omega)$; the kinetic energy of these solutions decays with time, and a result of non uniqueness of solutions is proved (see Theorem 5.2 below about uniqueness). See [4] for a broad discussion of those and other issues concerning the Euler equations and fluid mechanics more generally.

6.2. (S)-Weak solutions on \mathbb{T}^2 . We consider the Euler system (P) on $\mathbb{T}^2 = [0, 1]^2$, replacing non-penetrating boundary conditions with periodic boundary conditions. We are interested in deriving a counterpart of Theorem 5.2, in the case $p = \infty$ in this setting. The relevant space for the Euler data is then

$$\mathbf{V}^{1,\infty}(\mathbb{T}^2) = \{\mathbf{u} \in \mathbf{L}^\infty(\mathbb{T}^2) : \text{curl } \mathbf{u} \in L^\infty(\mathbb{T}^2), \text{div } \mathbf{u} = 0 \text{ on } \mathbb{T}^2, \mathbf{u}(\cdot, 0) = \mathbf{u}(\cdot, 1), \mathbf{u}(0, \cdot) = \mathbf{u}(1, \cdot)\}$$

In this setting, there is no need to smoothen up the domain, and one obtains smooth approximating solutions $\mathbf{u}^{(n)}$ originating from smooth data $\mathbf{u}_0^{(n)}, \mathbf{f}^{(n)}$ approximating respectively \mathbf{u}_0 and \mathbf{f} as in (5.12). The same scheme of Sections 4.2 and 5 can be devised to obtain the following result.

Theorem 6.1. *Let $\mathbf{u}_0 \in \mathbf{V}^{1,\infty}(\mathbb{T}^2)$, $\mathbf{f} \in L^\infty(0, T; \mathbf{V}^{1,\infty}(\mathbb{T}^2))$ be given. Then there exists a unique (S)-Weak solution to the Euler system (P) on \mathbb{T}^2 , with periodic boundary condition, and data \mathbf{u}_0, \mathbf{f} ; that is, a pair $(\mathbf{u} : \Omega \times [0, T] \rightarrow \mathbb{R}^2, \pi : \Omega \times (0, T) \rightarrow \mathbb{R})$ with $\mathbf{u}(0) = \mathbf{u}_0$ and*

$$(6.2) \quad \mathbf{u} \in W^{1,\infty}([0, T]; \mathbf{BMO}(\mathbb{T}^2)) \cap L^\infty(0, T; \mathbf{V}^{1,\infty}(\mathbb{T}^2)), \quad \mathbf{u}(t) \in \mathbf{V}^{1,\infty}(\mathbb{T}^2) \quad \forall t \in [0, T],$$

$$(6.3) \quad \partial_t \mathbf{u} \in L^\infty(0, T; \mathbf{BMO}(\mathbb{T}^2)), \quad \nabla \mathbf{u} \in L^\infty(0, T; \mathbf{BMO}(\mathbb{T}^2)^{2 \times 2})$$

$$(6.4) \quad \nabla \pi \in L^q(0, T; \mathbf{L}^q(\mathbb{T}^2)), \quad \forall q < \infty,$$

$$(6.5) \quad \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla \pi = \mathbf{f} \quad \text{in } L^q(0, T; \mathbf{L}^q(\mathbb{T}^2)) \quad \forall q < \infty \quad \text{and a.e. in } \mathbb{T}^2 \times (0, T).$$

Remark 6.1. Comparing the above Theorem to Theorem 5.2, the only improvement in regularity is that $\partial_t \mathbf{u}$ is uniformly in time bounded in $\mathbf{BMO}(\mathbb{T}^2)$ (which, due to John-Nirenberg's inequality, is stronger than being uniformly bounded in $\mathbf{L}^q(\Omega)$ for all $q < \infty$). This descends from the uniform boundedness of $\partial_t \mathbf{u}^{(n)}$, which is obtained in turn by modifying the proof of Lemma 4.5 as follows. It is well known that the Leray projector $P_{\mathbb{T}^2}$ is given by

$$(P_{\mathbb{T}^2} \mathbf{v})_j = \mathbf{v}_j + R_j R_1 \mathbf{v}_1 + R_j R_2 \mathbf{v}_2$$

where R_j stands for the j -th Riesz transform, which is bounded on $\mathbf{BMO}(\mathbb{T}^2)$ by standard Calderon-Zygmund theory. Therefore

$$\|P_{\mathbb{T}^2} \mathbf{v}\|_{\mathbf{BMO}(\mathbb{T}^2)}, \|P_{\mathbb{T}^2}^\perp \mathbf{v}\|_{\mathbf{BMO}(\mathbb{T}^2)} \lesssim \|\mathbf{v}\|_{\mathbf{BMO}(\mathbb{T}^2)}.$$

Arguing as in (4.33), we then have, for each fixed $t \in (0, T)$,

$$\|P_{\mathbb{T}^2}^\perp(\omega^{(n)} \mathbf{u}^{(n)})\|_{\mathbf{BMO}(\mathbb{T}^2)} \lesssim \|(\omega^{(n)} \mathbf{u}^{(n)})\|_{\mathbf{BMO}(\mathbb{T}^2)} \leq \|\omega^{(n)} \mathbf{u}^{(n)}\|_{\mathbf{L}^\infty(\mathbb{T}^2)} \leq \mathcal{Q}(\mathbf{u}_0, \mathbf{f}).$$

Carrying on with the proof of Lemma 4.5 with no further changes implies the uniform bound

$$\|\partial_t \mathbf{u}^{(n)}\|_{L^\infty(0, T; \mathbf{BMO}(\mathbb{T}^2))} \lesssim \|(\omega^{(n)} \mathbf{u}^{(n)})\|_{\mathbf{BMO}(\mathbb{T}^2)} \leq \|\omega^{(n)} \mathbf{u}^{(n)}\|_{\mathbf{L}^\infty(\mathbb{T}^2)} \leq \mathcal{Q}(\mathbf{u}_0, \mathbf{f})$$

as claimed.

6.3. Theorem 5.2 for more general domains with corners and open problems. We wish to discuss a more general class of domains, of which the rectangle $[0, L_1] \times [0, L_2]$, object of Theorem 5.2, is possibly the simplest instance. We say that $\Omega \subset \mathbb{R}^2$ is a *polygonal-like domain* if it is a bounded simply connected open set and $\partial\Omega$ is a piecewise \mathcal{C}^2 planar curve, with finitely many points $\{S_1, \dots, S_N\}$ of discontinuity for the tangent vector $\mathbf{n}(x)$, $x \in \partial\Omega$; we call Γ_j the component of $\partial\Omega$ with end points S_j, S_{j+1} (and $S_{N+1} = S_1$). Then the tangent and normal vectors $\mathbf{t}_j, \mathbf{n}_j$ to Γ_j belong to $\mathcal{C}^1(\Gamma_j)$, for each $j = 1, \dots, N$, and we write

$$\mathbf{n}_-(S_j) = \lim_{\substack{x \rightarrow S_j \\ x \in \Gamma_{j-1}}} \mathbf{n}_j(x), \quad \mathbf{n}_+(S_j) = \lim_{\substack{x \rightarrow S_j \\ x \in \Gamma_j}} \mathbf{n}_j(x).$$

We assume for definiteness that $\mathbf{n}_-(S_j) \neq \mathbf{n}_+(S_j)$ and set

$$(6.6) \quad \alpha_j := \text{angle}(\mathbf{n}_-(S_j), \mathbf{n}_+(S_j)), \quad j = 1, \dots, N, \quad \underline{\alpha} := \min\{\alpha_j\}, \quad \bar{\alpha} := \max\{\alpha_j\}.$$

For this class of domains, under the additional assumption that $\bar{\alpha} \leq \frac{\pi}{2}$, given $1 < p < \infty$ and $f \in L^p(\Omega)$, the unique solution $G_\Omega f \in H_0^1(\Omega)$ belongs to $W^{2,p}(\Omega)$ as well and satisfies the estimate

$$(6.7) \quad \|G_\Omega f\|_{W^{2,p}(\Omega)} \leq C_{p,\Omega} \|f\|_{L^p(\Omega)}.$$

If it is the case that $\alpha_j > \frac{\pi}{2}$ for some j , a singular part is present and analogues of (6.7) only hold in a range of exponents $1 < p < p_0$, with $p_0 > 2$ depending on $\bar{\alpha}$. These results are taken from Sections 4.1-4.4 of Grisvard's classical treatise [19]. See also [20]. Tracking the proof in [19], one finds that dependence on p of the constant $C_{p,\Omega}$ in (6.7) as $p \rightarrow \infty$ is $O(p^2)$, in contrast to the case of smooth domains, where $C_{p,\Omega} = O(p)$. The appearance of an extra power of p is intrinsic in Grisvard's proof, which reduces the study of the Dirichlet problem on the corner with aperture α to an elliptic problem on the strip $\mathbb{R} \times (0, \alpha)$. The (second derivatives) of the correspondent Green's functions on the strip are Calderón-Zygmund kernels only separately in each variable, and thus fall into the scope of product Calderón-Zygmund theory. In general, the L^p norm of a product Calderon-Zygmund operator is allowed to grow at order p^2 as $p \rightarrow \infty$, and the relevant endpoint space is no longer BMO, but the smaller product BMO($\mathbb{R} \times (0, \alpha)$). See [7] for references on this point.

A consequence of our analysis in Section 3, in particular, of Proposition 3.1, is that $C_{p,\Omega} = O(p)$ in the particular case of a rectangular domain: this is a consequence of (3.4) and John-Nirenberg's inequality. Our analysis does not extend to a general polygonal-like domain with acute corners (where an estimate of the type (6.7) for all $p > 2$ is allowed to hold). However, estimate (3.4) can be shown to hold also for polygonal-like domains where all angles are of the type $\alpha_j = \frac{\pi}{m(j)}$ for some integer $m(j) \geq 2$. The proof is analogous: after localization and a suitable change of coordinates which straightens up the boundary, one studies the resulting (general) elliptic problem in the corner with aperture α_j , and applies an odd reflection argument to reach the halfspace. The proof is finished by applying the analogue of Theorem 3.1 for a general elliptic operator of class \mathcal{C}^2 . Thus, for a polygonal-like domain with angles of the type $\alpha_j = \frac{\pi}{m(j)}$, $C_{p,\Omega} = O(p)$ in (6.7).

As shown in the proof of Theorem 5.2, having $C_{p,\Omega} = O(p)$ in (6.7) is sufficient to get uniqueness of solutions with bounded initial vorticity to the Euler system (P) on Ω . Hence, Theorem 5.2 holds true *verbatim* if we replace the rectangle $[0, L_1] \times [0, L_2]$ with a domain of this sort. However, the proof of part (a) of Theorem 5.2 only relies on the same techniques

used for Theorem 5.1 and on the elliptic estimate (6.7). Therefore, this part of Theorem 5.2 holds for all polygonal-like domains with $\bar{\alpha} \leq \frac{\pi}{2}$. It is an open question whether part (b) of Theorem 5.2 holds for this class of domains as well.

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